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Effective Performance of Bessel Arrays

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The Bessel array is a configuration of 5, 7, or 9 identical speakers in an equal-spaced line array that provides the same overall polar pattern as a single speaker of the array. This study reports the results of a computer simulation, using point sources, to determine the effective operating frequency range, working distance, efficiency, power handling, maximum acoustic output, efficiency-bandwidth product, and power-bandwidth product, etc., of the array. The various Bessel configurations are compared to 1, 2, and 5 source equal-spaced equal-level equal-polarity line arrays.

As compared to a two-source equal-level in-phase array, a five source Bessel array is 2.4 dB less efficient, can handle 1.75 (+2.4 dB) more power, has the same maximum midband acoustic output power, and is usable for omnidirectional radiation 10 times higher in frequency! A working distance of twenty times the length of the Bessel array was assumed, with the length of the Bessel array (center-to-center measurement) being four times that of the two-source array.

Analysis reveals that the three Bessel arrays have equal maximum acoustic output, but that the five-element Bessel array has the highest efficiency and power-bandwidth product. The 7 and 9-source Bessel arrays are found to be effectively unusable, as compared to the 5-source Bessel, due to much lower efficiency, requirement for more sources, and poor high-frequency performance.

Judging polar peak-to-peak ripple and high frequency response, the performance of the Bessel array is found to improve in direct proportion to working distance away from the array. Unfortunately, the phase vs direction and phase vs frequency characteristics of the Bessel array are very non-linear and make it difficult to use with other sources.

0. INTRODUCTION

The Bessel array is a patented [1, assigned to U. S. Philips Corporation] configuration of equally-spaced identical transducers that is said to provide the same overall polar pattern as the polar pattern of a single transducer of the configuration. It is a method that extends the directional operating bandwidth of an array of transducers up into the region where the length of the array is a large number of wavelengths [2], [3]. The introduction to the Philips article describes the justification for the Bessel configuration [2]:

"An array of N loudspeakers connected in parallel and in phase can radiate N^2 times as much power as a single loudspeaker at very low frequencies, but only N times as much at high frequencies. The power response of the array is therefore quite different from that of the single loudspeakers that compose it. This is due to the increased directivity of the array; whereas the radiation pattern of a single loudspeaker is reasonable omnidirectional, usually up to at least a few kilohertz, that of an array is so only at low frequencies. At high frequencies it becomes much more directive; moreover, the directivity varies considerably with frequency. ... These shortcomings can be remedied, at some expense to power radiation, by correctly proportioning the drive to the individual speakers of the array. The required proportioning coefficients are based on the Bessel functions."

The configuration normally takes the form of a five, seven, or nine- element line array or a twenty five (5 x 5) element symmetrical planar (panel) source. Only the line array Bessel configurations are analyzed in this study. The method used to set the drive levels of the transducers in the array essentially randomizes the polarity of each of the elements [4], [5]. These polarity reversals dramatically reduce the sensitivity and efficiency of the resultant array as compared to an equal-drive-level equal-polarity array. However, the chosen drive levels do dramatically extend the directional operating bandwidth of the array up into the region where the array is many wavelengths long.

To my knowledge, most (if not all) of the available references to the Bessel array contain hardly any information on the effective operation of the configuration. Some questions that immediately come to mind are: How high in frequency does the array operate? How far away from the array must you be? How does the efficiency, power handling, and maximum acoustic output compare to other array configurations? Which of the three array types: five, seven, or nine-element Bessel, is the best?

These and other questions are answered in this paper by analyzing the Bessel configuration using simulations based on arrays of point sources. The point source, being omnidirectional, should provide omnidirectional radiation when arrayed in a Bessel configuration. The degree to which the analyzed configurations provide omnidirectional coverage is the basis for evaluating their effective performance.

1. REVIEW OF BESSEL DERIVED SOURCE LEVELS

Quoting again from [2]:

"Consider an array of $2N+1$ speakers equidistantly spaced in a straight line and driven by a common signal multiplied by coefficients $(a_{-N}, a_{-N+1}, \dots a_0, \dots a_{N-1}, a_N)$ peculiar to each speaker. Assume that

- the point of observation P is in the far-field region of each speaker

- the radiation of each speaker is not influenced by the others

- all speakers have the same frequency and directional response $A(\omega,\theta)$."

The required proportioning of drive levels (both level and polarity) of each of the transducers of the configuration is based on numbers derived from the Bessel function of first kind and order n [2], [6]:

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\,(n+k)!} \tag{1}$$

The method relies on a mathematical property of the Bessel function which is:

$$\left|\sum_{n=-\infty}^{\infty} J_n(z)\right| = \left|\sum_{n=-\infty}^{\infty} J_n(z)e^{(jnx)}\right| = \left|e^{(jz\sin x)}\right| = 1 \qquad (2)$$

This property, combined with the equation that gives the pressure magnitude and phase at point P, at a particular frequency and angle for an array of sources (at an infinite distance):

$$p(\omega,\theta) = A(\omega,\theta)\sum_{n=-N}^{N} a_n e^{(-jnx)} \qquad (3)$$

where

$$x = \frac{\omega l \sin\theta}{c}, \text{ (assumes sample point at infinite distance)}$$

$\omega =$ frequency, radians per sec $= 2\pi f$

$c =$ velocity of sound

$l =$ distance between speakers

$\theta =$ angle between sample point vector and array axis

$A(\omega,\theta) =$ amplitude - phase function that gives the directional characteristics of a single source

$a_n =$ drive level of source n giving strength and polarity

yields a function that makes the dependence of the magnitude of p on direction and frequency the same for the array as for a single speaker that makes up the array:

$$p(\omega,\theta) = A(\omega,\theta)\sum_{n=-N}^{N} J_n(z)e^{(-jnx)}$$

$$= A(\omega,\theta)e^{(-jz\sin x)} \qquad \text{if } N \to \infty$$

or

$$|p(\omega,\theta)| = |A(\omega,\theta)| \qquad (4)$$

The last equation clearly shows that the polar pattern of the array will be same as one of the sources that makes up the array. This function only works exactly, of course, for an infinite array of sources and a sample point at an infinite distance from the array. A finite sized array of 5, 7, or 9 sources is also found to work quite well even if the drive levels are restricted to approximate values limited to the integer ratios ±1 and ±2 (±0.5 and ±1 in practice). These approximate values allow the drive levels of the array to be set by simple series/parallel connections of the drivers.

The approximate coefficient values are derived from the Bessel function by searching for arguments (both integer and non-integer values of z are allowed) that yield a coefficient ratio series that can be approximated by ±1 and ±2. An argument value of $z = 1.5$ is found to be a good choice for the 5 element array coefficients. Fig. 1 shows the resultant coefficient values of $J_n(1.5)$, over the range $-10 <= n <= +10$, plotted in bar graph form. Both the actual values and the absolute values are plotted for comparison purposes. The plotted values show that the function very rapidly decreases to very small values for n beyond ±3. Truncating the series beyond these values eliminates relatively little from the sum.

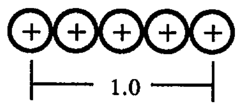
Choosing the range -2<= n <= 2 yields the series:

$$J_{-2}(1.5),\ J_{-1}(1.5),\ J_0(1.5),\ J_2(1.5),\ J_3(1.5) =$$
$$0.232,\ -0.558,\ 0.512,\ 0.558,\ 0.232$$

whose ratios can be approximated by

Five element Bessel ratios = +0.5 : -1 : +1 : +1 : +0.5.

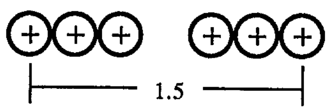
Configuration:



Likewise, the corresponding series for $z = 2.405$, and $z = 3.83$ yield the approximate drive ratios for the 7 and 9 element Bessel arrays respectively:

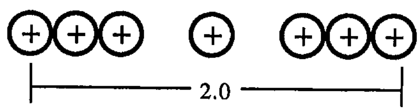
Seven element Bessel ratios = -0.5 : +1 : -1 : 0 : +1 : +1 : +0.5

Configuration:



Nine element Bessel ratios = +0.5 : -1 : +1 : 0 : -1 : 0 : +1 : +1 : +0.5

Configuration:



Note that sources that have zero drive levels can be eliminated from their respective arrays, making the 7 element Bessel array have 6 actual sources and the 9 element Bessel array have 7 actual sources. Note also, that the spaces for the removed sources must be preserved.

These drive ratios can be implemented by simple series/parallel hookups for each of the three configurations. Each ratio combination can be connected in a mostly-parallel or mostly-series hookup. Only the more practical mostly-parallel connection will be analyzed here.

2. SIMULATION METHODS

The polar and frequency response simulations were accomplished by evaluating a more complete version eq. (3), that takes proper account of the actual distance from each source to the sample point, with no approximations. This more complete equation allows proper evaluation of effective working distances from the array. The equation appears as:

$$p(\omega,\theta) = A(\omega,\theta)\sum_{n=-N}^{N}\frac{a_n}{r_n}e^{(-jkr_n)} \tag{7}$$

where

$k =$ wave number $= \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$

$\omega =$ frequency, radians per sec $= 2\pi f$

$c =$ velocity of sound

$\lambda =$ wavelength $= \frac{c}{f}$

$f =$ frequency, Hz

$r_n =$ distance from source n to sample point $= |\mathbf{r_n}|$

$a_n =$ strength and polarity of source n

All distances in this paper are referenced to a system that has a unit velocity of transmission. This means that at a frequency of one Hz a unit distance is one wavelength long. All the 5 element arrays have a unit overall source center-to-center length. A working distance of 20 units implies that the pressure sampling point is 20 times the length of the 5 element array away from the center of the array.

The array was oriented so that its axis was along the y axis, with its center at the origin of the coordinate system. Rotation was always around the center of the array (the origin), with the zero angle in the direction of the positive x axis and positive angles in the counter-clockwise direction. For the Bessel arrays, the sources with higher number Bessel coefficients were above the x axis (positive angles).

A number of the analysis factors require the calculation of the peak-to-peak ripple in dB for the polar directional pattern at a specific frequency. The equation to calculate this ripple factor [2] appears as:

$$\text{Peak - to - Peak Ripple dB} = R(\omega) = 20\log\left(\frac{|p(\omega,\theta)|_{\max}}{|p(\omega,\theta)|_{\min}}\right)_{-\pi\le\theta\le\pi} \tag{8}$$

All calculations for this paper were done on a Macintosh II desktop computer using a combination of the Microsoft spreadsheet program "Excel™" and the math analysis program "Mathematica™" by Wolfram Research, Inc (used for all graphic output). All equations were set using the Macintosh program "MathType™"by Design Science, Inc.

3. ARRAY ANALYSIS FACTORS

The various Bessel configurations (five, seven, and nine-source line arrays) were compared to one, two, and five-source equal-spaced equal-level equal-polarity line arrays. A number of analysis factors were used in the comparison: voltage sensitivity, impedance, efficiency, maximum input power handling, maximum acoustic output power and SPL, maximum operating frequency, working distance, polar directional response, frequency response (both magnitude and phase), polar peak-to-peak ripple vs frequency, efficiency-bandwidth product, power-bandwidth product, and power-bandwidth product per unit. These analysis factors are individually described in the following subsections:

3.1. Voltage Sensitivity

The voltage sensitivity of a system is the on-axis sound pressure level (SPL) generated at a specific distance for a particular applied voltage. In this paper, all measurements are referenced or normalized to a point source which is assumed to have all unit specifications ie., a sensitivity of 1 or 0 dB SPL for an applied unity voltage, at a one unit distance. The sensitivity of the analyzed arrays is just simply the total of the individual drive levels.

3.2. Impedance

The input electrical impedance for each analyzed array was computed assuming a unity impedance (resistance) for each of the individual sources.

3.3. Efficiency

The electro-acoustical efficiency (electrical input power divided by the resultant acoustical output power) of each array was computed by direct comparison to a single point source, in the omnidirectional radiation region of the array's frequency range. An efficiency of unity was assigned to the point source. The efficiency of an array was computed by squaring its sensitivity and multiplying by its impedance:

$$\eta_o = \frac{P_{out}}{P_{in}} = \text{Sens}^2 \ Z_{in} \tag{5}$$

3.4. Power Handling

The maximum input electrical power handling of an array was computed by summing the individual source powers computed by applying a unity input voltage and assuming unity impedances for all individual sources.

3.5. Maximum Acoustic Output Power and SPL

The maximum acoustic output power was computed by multiplying the array's efficiency by its maximum input electrical power:

$$P_{out} = \eta_o \, P_{in} \tag{6}$$

The sound pressure level in dB was calculated by using $10Log_{10}(P_{out}/P_{ref})$, where P_{ref} is unity (the power output of a single source).

3.6. Maximum Operating Frequency

The maximum operating frequency was assessed by simulating the polar response of the analyzed array using eq. (7) and then finding the maximum frequency up to which the peak-to-peak polar magnitude ripple (eq. (8)) did not exceed a specific amount, usually 3, 4, 6, or 9 dB. Note that a point source has a maximum operating frequency of infinity, using this definition!

3.7. Working Distance

The working distance was assessed similarly to maximum operating frequency by polar simulations and then noting the minimum operating distance that provided a specific peak-to-peak polar ripple. Usually a specific working distance (in array length terms: 5, 10, or 20 units) was chosen and then all the relevant parameters were calculated.

3.8. Polar Response

Polar directional responses were computed using equation (7) at various angles and distances for each of the analyzed configurations. A set of polar responses at a fixed working distance (usually 20 units) at various frequencies were simulated along with a set at a fixed frequency (usually 10 Hz) at various working distances. Both magnitude and sometimes phase vs angle plots are displayed. The linear phase effects of transport delay between source and sample point were removed in all phase displays.

3.9. Frequency Response (Magnitude and Phase)

Magnitude and phase frequency responses were computed using equation (7) at various angles and distances for each of the analyzed configurations. A set of frequency responses at various angles at a fixed distance (usually 20 units) were simulated along with a set at a fixed angle (usually 45 degrees) at various working distances. Both magnitude and sometimes phase (also group delay in one case) vs frequency plots are displayed. The linear phase effects of transport delay between source and sample point were removed in all phase displays.

3.10. Polar Peak-To-Peak Ripple vs Frequency

A plot of the polar peak-to-peak ripple in dB vs frequency indicates the tradeoff of polar non-linearities vs high frequency limit. In general, all arrays exhibit increasing polar ripple as the frequency is increased.

3.11. Efficiency-Bandwidth Product

The efficiency-bandwidth product was computed by forming the product of the efficiency and the maximum operating frequency. This number gives you a comparative value that indicates how thrifty the analyzed array is in terms of its efficiency and operating frequency range.

3.12. Power-Bandwidth Product

The power-bandwidth product was computed by forming the product of the maximum acoustic output power and the maximum operating frequency. This number gives you a comparative value that indicates how how well the analyzed array functions in terms of its output power and operating frequency range.

3.13. Power-Bandwidth Product per Unit

The power-bandwidth product per unit was computed by dividing the power-bandwidth product by the number of units in the array. This number can be thought of as a figure of merit for comparing the operating effectiveness of the analyzed arrays, on a per-unit basis.

4. SIMULATION RESULTS

Several different point source configurations were analyzed and compared for this study. All configurations were analyzed in terms of the performance of a single point source. The configurations analyzed are described as follows:

1) ***Two equal-level equal-polarity sources*** with center-to-center ***spacing of 0.25*** (same center-to-center spacing as the individual spacing of the five element arrays),
2) ***Two equal-level equal-polarity sources*** with center-to-center ***spacing of 1.0*** (same overall center-to-center length as the five element arrays),
3) ***Five equal-level equal-polarity equal-spaced sources*** with individual center-to-center spacing of 0.25 and overall center-to-center ***length of 1.0***,
4) ***Five source Bessel array*** with individual center-to-center spacing of 0.25 and overall center-to-center ***length of 1.0***,
5) ***Seven source Bessel array*** with individual center-to-center spacing of 0.25 and overall center-to-center ***length of 1.5***, and
6) ***Nine source Bessel array*** with individual center-to-center spacing of 0.25 and overall center-to-center ***length of 2.0***.

Note that all the arrays have the same individual center-to-center spacing (except for the two source configuration with center-to-center length of 1.0). This means that the overall array length increases in direct proportion to the number of sources. This models the real world situation of using the same size transducers packed as close together as possible.

For each configuration, several possible analysis factors were calculated: voltage sensitivity, impedance, efficiency, maximum input power handling, maximum acoustic output power and SPL, maximum operating frequency, working distance, polar directional response, frequency response (both magnitude, phase, and group delay), efficiency-bandwidth product, power-bandwidth product, and power-bandwidth product per unit. See previous section 2 for further explanations of these factors.

The results of the simulations are described in the following sections and shown in Figs. 2 to 32.

4.1. Single Point Source

A single point source is the reference for all the following array configurations. The single point source is arbitrarily assigned all unit parameters and its characteristics are shown in Table 1. Note that all the frequency dependent factors have infinite values because the point source by definition has no upper frequency limit.

The polar response of the point source (not shown) is a perfect circle, while its frequency and phase responses (not shown) are straight lines. The polar and frequency responses of the reference point source are not distance dependent. Note that table entries have been reserved for working distances of 5, 10, and 20 units at peak-to-peak ripple values of 3, 4, and 6 dB.

4.2. Two Sources, Equal-Level Equal-Polarity with 0.25 and 1.0 C-C Spacing

The two-source array is the simplest configuration one step above the single source, and is used quite frequently to increase the acoustic output, as compared to a single source. Unfortunately, as the following simulations show, the maximum frequency of operation drops dramatically because of source interference and lobing. Two double-source configurations, with center-to-center spacing of 0.25 and 1.0 units, were analyzed and are described in the following section.

The two-source array with 0.25 unit c-c spacing has the same center-to-center spacing as the individual spacing of the five element arrays. This close side-by-side spacing is the logical configuration for getting the most performance (highest operating bandwidth) from a two source array. All the characteristics and calculated parameters for the two-source array with 0.25 c-c spacing are shown in Table 2.

The 1.0 unit c-c spaced two-source array has the same center-to-center spacing as the overall c-c spacing (outside sources) of the five element arrays. If you just simply remove the center three sources of the five element array you get this spacing. All the characteristics and calculated parameters for the two-source array with 1.0 c-c spacing are shown in Table 3.

All the responses and characteristics for the 1.0 unit c-c spacing array are the same as the 0.25 unit c-c spacing array but shifted down in frequency by two octaves (frequency x 1/4). The data on the 1.0 unit c-c spacing array has been included for comparing against the five-source arrays, which have the same length.

4.2.1. Polar Responses

The polar magnitude responses of the two-source array with 0.25 c-c spacing, at constant distance, are shown in Fig. 2. The polars are displayed at half-decade intervals from 0.316 Hz to 31.6 Hz and at a working distance of 20 units. An additional polar at 2.0 Hz is also displayed. All the polar plots displayed in this paper cover a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB. Note that at 2 Hz, where the sources are one-half wavelength apart, the first polar null at 90 degrees off axis occurs. Note also, that above about 1.8 Hz, the polar response is multilobed and hence unusable for omnidirectional response.

The polar responses for the 0.25 c-c spacing array, at a fixed frequency of 10 Hz and at different working distances, are shown in Fig. 3. Polars at distances of 1.25, 2.5, 5, 10 and 100k units are shown. Observe that the polar responses essentially exhibit no change with increasing working distance beyond about 2.5 units (10 times array length). Note that the 100k (one hundred thousand) unit distance is extremely far from the array; essentially an effective infinity. If the overall length of the array were 2 ft (0.6 m), this distance would be about 38 miles (60 km) away!

Only a few polar responses were done on the two-source array with 1.0 c-c spacing, mainly to illustrate the variation of phase vs angle with frequency and working distance. The previous two-source array exhibits the same behavior but four times higher in frequency. Note that the first polar null at 90 degrees off axis occurs at a frequency of 0.5 Hz where the sources are one-half wavelength apart (not shown).

Fig. 4 shows various magnitude and phase polar responses at different frequencies (*f*) and distances (*D*) for the two-source array with 1.0 c-c spacing. The following four combinations are plotted:

1) $f = 1$ Hz, $D = 20$ units;
2) $f = 1$ Hz, $D = 100$k;
3) $f = 2$ Hz, $D = 20$;
4) $f = 2$ Hz, $D = 100$k.

The phase vs direction plots show the phase of the pressure at the sample point vs the off-axis direction. The phase values are referenced to the input signal of the array. The effects of linear phase lag and delay due to sample distance have been removed in this plot and in all the polar and frequency response plots of this paper. Note that the phase values switch between 0 and ±180 degrees depending on which polar lobe the pressure sample point happens to be on. The phase always starts out at zero degrees (on axis). At distances far from the array, the phase transitions occur much abruptly with angle, with no rounded corners.

4.2.2. Frequency Responses

The magnitude vs frequency responses of the two-source array with 1.0 c-c spacing at constant distance are shown in Fig. 5. The responses are shown at angles ranging from 0 to +90 degs with steps of 15 degs at a working distance of 20 units. Note that the response progressively gets rougher as the angle increases due to the nulls in the response moving down in frequency. The magnitude vs frequency responses at a fixed angle of 45 degs and at different working distances are shown in Fig. 6. Distances of 1.25, 2.5, 5, 10, and 100k units are shown. Note that the frequency response changes very little with distance beyond 5 units.

The frequency range of the responses goes from 0.1 to 10 Hz with a log frequency scale. Note that the frequency scale is marked with decade number (log f) rather than frequency (-1.0 = 0.1 Hz, 0.0 = 1 Hz, etc).

To illustrate the variation of phase vs frequency and working distance, several magnitude and phase responses were done on the two-source array with 1.0 c-c spacing. Fig. 7 shows these responses with a fixed angle of 45 degs and a distance of 5 and 100k units. Observe that the phase again is either 0 or ±180 degrees, depending on which polar lobe the sample point happens to be in. This phase vs frequency behavior looks suspiciously non-minimum phase but is actually minimum-phase [7]. This comment only applies to the two-source array, however, where the response at the sample point is strictly due to a signal plus a single delayed signal of reduced amplitude.

4.2.3. Polar Peak-To-Peak Ripple vs Frequency

Fig. 8 shows a plot of the polar peak-to-peak ripple in dB vs frequency for both two-source arrays at a working distance of 20 units. Note that the ripple increases very rapidly above 0.25 Hz for the 1.0 unit spacing and above 1 Hz for the 0.25 unit spacing.

Fig. 9 shows the polar peak-to-peak ripple vs frequency at several different working distances from 2.5 to 100k units for the 1.0 spaced two-source array. The graph exhibits essentially no change at distances beyond 5 units.

Fig. 10 shows a polar response of the 0.25 spaced array at a distance of 20 units and a frequency of 1.1 Hz, which corresponds to the frequency where the peak-to-peak ripple is 4 dB. Note that the polar is very smooth, but squashed vertically, and exhibits its maximum deviation (-4 dB) at ±90 degs. As will be seen, this is a characteristic of all the equal-level equal-phase arrays.

The following approximate equations relate the maximum operating frequency for omnidirectional radiation (f_{max}) to the array length, for the two-source equal-level equal-polarity arrays:

For the 0.25 unit spaced array:

$$
\begin{aligned}
f_{max} &\approx 1.0\frac{c}{L} && \text{for 3 dB peak - to - peak polar ripple} \\
&\approx 1.1\frac{c}{L} && \text{for 4 dB peak - to - peak polar ripple} \\
&\approx 1.3\frac{c}{L} && \text{for 6 dB peak - to - peak polar ripple} \\
&\approx 1.6\frac{c}{L} && \text{for 9 dB peak - to - peak polar ripple}
\end{aligned}
\qquad (7)
$$

For the 1.0 unit spaced array:

$$
\begin{aligned}
f_{max} &\approx 0.25\frac{c}{L} && \text{for 3 dB peak - to - peak polar ripple} \\
&\approx 0.28\frac{c}{L} && \text{for 4 dB peak - to - peak polar ripple} \\
&\approx 0.33\frac{c}{L} && \text{for 6 dB peak - to - peak polar ripple} \\
&\approx 0.40\frac{c}{L} && \text{for 9 dB peak - to - peak polar ripple}
\end{aligned}
\qquad (8)
$$

where

$c =$ velocity of sound

$L =$ length of array (center - to - center distance of sources)

4.2.4. Discussion

At low frequencies, the two-source arrays exhibit mostly omnidirectional behavior below 0.25 Hz for the 1.0 unit spaced array and below 1.0 Hz for the 0.25 unit spaced array. The upper frequency limit for omnidirectional radiation occurs at the frequency where the sources are about one-quarter wavelength apart. Below this frequency, the efficiency is twice that of the single source while the maximum output is four time the single source.

The directional characteristics essentially do not change with working distance beyond a point that is roughly 10 times the length of the array. The behavior of the unit spaced two-source array exhibits the same activity as the 0.25 spaced two-source array but at one-fourth the frequency.

The off-axis polar phase alternates between 0 and 180 degs depending on what polar lobe the sample point is in. The off-axis phase vs frequency data exhibits the same switching behavior with increasing frequency but is found to be minimum phase.

4.3. Five Sources, Equal-Level Equal-Polarity Equal-Spaced with Overall C-C Length of 1.0

This array contains five sources equally spaced with equal levels and equal polarities. The overall length, measured from the centers of the outside sources, is one unit. The individual source center-to center spacing is 0.25 units. The characteristics and calculated parameters for this array are shown in Table 4.

This array provides 25 times the acoustic output power at low frequencies as compared to a single source. This array was included for direct comparison to the 5 source Bessel array. The only difference between this array and the Bessel array is the amplitude and polarity of the source drive levels.

4.3.1. Polar Responses

The polar magnitude responses of the five-source equal-level array, at constant distance, are shown in Fig. 11. The polars are displayed at half-decade intervals from 0.1 Hz to 10 Hz at a working distance of 20 units. The polar plot covers a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB. The polars of this array are much more complex and directive than the two-source polars (Fig. 2). For omnidirectional radiation, the five-source equal-level array is unusable above about 0.6 Hz.

The polar magnitude responses for the five-source equal-level array, at a fixed frequency of 10 Hz and at different working distances, are shown in Fig. 12. Polars at distances of 1.25, 2.5, 5, 10, 20 and 100k units are shown. Note that the polar response changes very little with distance beyond roughly 10 units (10 array lengths).

Fig. 13 shows various magnitude and phase polar responses at different frequencies (*f*) and distances (*D*) for the five-source array. The following five combinations are plotted:

1) $f = 0.5$ Hz, $D = 20$ units;
2) $f = 1$ Hz, $D = 20$;
3) $f = 1$ Hz, $D = 100$k;
4) $f = 2$ Hz, $D = 20$;
5) $f = 2$ Hz, $D = 100$k

The phase vs direction plots show the phase of the pressure at the sample point vs the off-axis direction. The effects of the linear phase delay due to sample distance have been eliminated. Note that the phase values switch between 0 and ±180 degrees depending on which polar lobe the pressure sample point happens to be on. The phase always starts out at zero degrees (on axis). At distances far from the array, the phase changes occur more abruptly with angle. The phase variation with direction for the five-source array is very similar to the behavior of the two-source arrays.

4.3.2. Frequency Responses

The magnitude vs frequency responses of the five-source equal-level array at constant distance are shown in Fig. 14. The responses are shown at angles ranging from 0 to +90 degs with steps of 15 degs at a working distance of 20 units. Note that the response progressively gets rougher as the angle increases, similarly to the two-source arrays.

The magnitude vs frequency responses at a fixed angle of 45 degs and at different working distances are shown in Fig. 15. Distances of 1.25, 2.5, 5, 10, and 100k units are shown. Note that the frequency response changes very little with distance beyond about 5 units.

The phase vs frequency behavior of the five-source array is shown in Fig. 16 where responses at 45 degs off axis at distances of 20 and 100k units are shown. The effects of linear phase lag and delay due to sample distance have been eliminated. The phase activity vs frequency is very similar to the two-source arrays, but is highly likely to be non-minimum phase due to the existence of the additional sources. The phase toggles rapidly between 0 and ±180 degrees as frequency increases.

4.3.3. Polar Peak-To-Peak Ripple vs Frequency

Fig. 17 exhibits a plot of polar peak-to-peak ripple in dB vs frequency for the five-source equal-level array at a working distance of 20 units. Note that the ripple increases very rapidly above 0.35 Hz. The five-source array has somewhat better performance than the two-source 1.0 unit array but significantly lower than the 0.25 unit two-source array (see Fig. 8). Additional data (not shown) indicates that the polar peak-to-peak ripple essentially does not change with working distances beyond about 10 units. This behavior is similar to the two-source arrays (see Fig. 9).

The following approximate equations relate the maximum operating frequency for omnidirectional radiation (f_{max}) to the array length, for the five-source equal-level equal-polarity array:

$$
\begin{aligned}
f_{\max} &\approx 0.35\frac{c}{L} && \text{for 3 dB peak-to-peak polar ripple} \\
&\approx 0.40\frac{c}{L} && \text{for 4 dB peak-to-peak polar ripple} \\
&\approx 0.48\frac{c}{L} && \text{for 6 dB peak-to-peak polar ripple} \\
&\approx 0.56\frac{c}{L} && \text{for 9 dB peak-to-peak polar ripple}
\end{aligned}
\tag{9}
$$

where

$c =$ velocity of sound

$L =$ length of array (center-to-center distance of outside sources)

Compare these multipliers to the previous values for the two-source arrays shown in eqs (7) and (8).

4.3.4. Discussion

At low frequencies, below about 0.35 Hz, the five-source array exhibits mostly omnidirectional behavior. The upper frequency limit for omnidirectional radiation occurs at the frequency where the length of the array is about one-third wavelength, which is somewhat higher than the two-element array. Below this frequency, the efficiency is five times that of the single source while the maximum output is 25 times higher.

The five-source equal-level array of unit c-c length operates slightly higher in frequency than the two-source equal-level array with 1.0 unit c-c spacing but significantly lower than the two-source equal-level array with 0.25 unit c-c spacing. The phase vs frequency curve is non-minimum phase. The phase vs angle and phase vs frequency curves alternate between 0 and ±180 degs.

The directional characteristics essentially do not change beyond a point that is roughly 10 times the length of the array (roughly the same as the two-source equal-level array).

4.4. Five-Source Bessel Array with Overall C-C Length of 1.0

The Bessel configuration is used to gain increased acoustic output without the severe narrowing directional characteristics with frequency exhibited by the equal-level equal-polarity equal-spaced line arrays. The Bessel array is said to have the same overall directional pattern as one of the sources that makes up the array.

The following simulation uses omnidirectional point sources to form the Bessel structure. The degree to which the overall polar response matches an omnidirectional pattern is used to judge the effectiveness of the Bessel array. The five source Bessel array contains the fewest number of sources of the three analyzed Bessel configurations. The characteristics and calculated parameters for the five-source Bessel array with 1.0 c-c overall length are shown in Table 5.

Because of the much greater upper frequency of the Bessel array, all the bandwidth product values are much higher than the previous arrays. However, the efficiency is only about 14% (+0.6 dB) greater than a single source. With the higher power handling of 3.5 times a single source, the maximum acoustic output of the Bessel array is the same as the two-source arrays.

4.4.1. Polar Responses

The polar magnitude responses of the five-source Bessel array, at constant distance, are shown in Fig. 18. The polars are displayed at half-decade intervals from 0.316 Hz to 100 Hz at a working distance of 20 units. The polar plot covers a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB.

Note the much greater high frequency range of operation as compared with the previous arrays. The polar ripple does not get significant until frequencies higher than about 10 Hz.

The polar magnitude responses for the five-source Bessel array, at a fixed frequency of 10 Hz and at different working distances, are shown in Fig. 19. Polars at distances of 1.25, 2.5, 5, 10, 20, 40, and 100k units are shown. Note that, unlike the previous arrays, the polar ripple appears to get smaller and smaller the farther away you get from the array! However, a limit of about 1.2 dB peak-to-peak ripple appears to exist even at the farthest distance. This figure is confirmed by [2, Table 2].

For omnidirectional radiation, at a distance of 20 units, with no more than 6 dB peak-to-peak ripple, the five-source Bessel array is usable up to beyond 18 Hz! As compared to an equal-level equal-polarity two-source array with c-c spacing equal to the overall c-c spacing of the Bessel array (0.33 Hz from Table 3), this represents an increase in upper frequency of about 55 times (≈18/0.33)!

A further study of the variation of polar ripple with distance was performed by simulating at a much higher frequency of 100 Hz (where the array length is 100 wavelengths!) and then varying the working distance from 10 to 1000 in three decade steps. Fig. 20 shows the results of these simulations. The polar at 10 units distance is unusable due to severe polar ripple (about 40 dB peak-to-peak), but settles down to about 2 dB p-p ripple at a distance of 1000 units (a long way away!). It appears that there is no effective upper limit to the frequency of operation of the Bessel array if you can get far enough away! Practically however, working distances in the range of 5 to 20 times the length of the array will define the operation of the array.

Fig. 21 shows a series of phase polar responses (phase vs direction angle) at a constant distance of 20 units at frequencies of 0, 0.1, 0.5, 1, 2, 4, 5, 10, and 20 Hz. The delay effects of the working distance have been compensated for thus making the on-axis phase zero in every case. Also shown is a phase polar response at a distance of 100k units at 20 Hz. The phase curves exhibit a highly-nonlinear sinusoidal like variation of phase with angle with a peak-to-peak amplitude of ±90 degs. For a fixed angular increment, the number of oscillation cycles increases with frequency.

4.4.2. Frequency Responses

The magnitude vs frequency responses of the five-source Bessel array, at constant distance, are shown in Fig. 22. The responses cover the range from 0.1 to 10 Hz, and are shown at angles ranging from 0 to +90 degs with steps of 15 degs, at a working distance of 20 units. Unlike the previous equal-level arrays, the ripple does not continually increase with angle.

The magnitude vs frequency responses at a fixed angle of 45 degs and at different working distances are shown in Fig. 23. Note the wider frequency range of 0.1 to 100 Hz. Responses at distances from 1.25 to 160 units with 1:2 steps are simulated, in addition to one at 100k units are shown. Note however, unlike the previous arrays, that the frequency response ripple continually decreases with distance until about a 2 dB p-p ripple is attained. This again reinforces the observation that the Bessel array performance can reach any arbitrary upper frequency if you move far enough away from the array.

The phase vs frequency behavior of the five-source Bessel array is shown in Fig. 24; where magnitude, phase, and group delay responses at 45 degs off axis at distances of 20 units are shown. Both log and linear frequency scale plots are shown in this graph, up to a frequency of 10 Hz. The phase varies non-linearly, in a somewhat sinusoidal manner with frequency, oscillating between ±90 degs. The magnitude response is mostly flat, with about a 2 dB p-p ripple, with more amplitude variations per unit frequency.

Because the magnitude is mostly flat and the phase varies dramatically with frequency, this magnitude-phase behavior vs frequency is highly non-linear and non-minimum phase. The group delay plot, shown in Fig. 24c, indicates an effective oscillatory peak shift of acoustic position of about ±25% the length of the array, as the frequency is increased! I am not going to venture an opinion on whether this is audible or not.

4.4.3. Polar Peak-To-Peak Ripple vs Frequency

Fig. 25 exhibits a plot of polar peak-to-peak ripple in dB vs frequency for the five-source Bessel array at a working distance of 20 units. Observe that the ripple increases much more gradually with increasing frequency as compared to the equal-level arrays. Note also the much extended bandwidth of operation as compared to the previous arrays. Also observe the plateau in the curve between 0.5 and 1.1 Hz where the ripple is about 1.3 dB. At a p-p ripple of 6 dB, operation extends up to a frequency of 18 Hz (line length of 18 wavelengths!).

To investigate the behavior of ripple with increasing distance, numerous plots of ripple vs frequency were done over the distance range of 2.5 to 100k units. This data is shown in Fig. 26. It is quite evident that the operation of the Bessel array improves in direct proportion to the working distance away from the array. Note that at points very far from the array, the p-p ripple attains a constant value of about 1.3 dB; this is the source of the plateau noted previously.

Fig. 27 shows a plot of maximum operating frequency vs operating distance for the five-source Bessel array. Contours of equal p-p ripple at values of 3, 6, and 9 dB are shown on the graph. The direct relationship between maximum frequency and operating distance is clearly shown. The contours of constant p-p ripple form straight lines on the graph except for slight deviations at small distances.

The following approximate equations relate the maximum operating frequency for omnidirectional radiation (f_{max}) to array length (L) and operating distance (D), for the five-element Bessel array. Note the dependence on distance, which was absent in the previous arrays equations (eqs (7)-(9)).

$$
\begin{aligned}
f_{\max} &\approx 0.40\frac{c}{L}D && \text{for 3 dB peak - to - peak polar ripple} \\
&\approx 0.55\frac{c}{L}D && \text{for 4 dB peak - to - peak polar ripple} \\
&\approx 0.90\frac{c}{L}D && \text{for 6 dB peak - to - peak polar ripple} \\
&\approx 1.4\frac{c}{L}D && \text{for 9 dB peak - to - peak polar ripple}
\end{aligned}
\qquad (10)
$$

where

$$D = \text{normalized operating distance} = \frac{d}{L}$$

$c =$ velocity of sound

$d =$ working distance away from center of array

$L =$ length of array (center - to - center distance of outside sources)

For comparison to the equations for the previous arrays (eqs (7)-(9)), the following equations evaluate eqs. (10) at a distance of 20 units:

$$
\begin{aligned}
f_{\max} &\approx 8\frac{c}{L} && \text{for 3 dB peak - to - peak polar ripple} \\
&\approx 11\frac{c}{L} && \text{for 4 dB peak - to - peak polar ripple} \\
&\approx 18\frac{c}{L} && \text{for 6 dB peak - to - peak polar ripple} \\
&\approx 28\frac{c}{L} && \text{for 9 dB peak - to - peak polar ripple}
\end{aligned}
\qquad (11)
$$

Note the large multipliers as compared to the previous arrays equations.

4.4.4. Discussion

The five-element Bessel array provides a very impressive increase in bandwidth of operation when compared to equivalent two and five-source equal-level equal-polarity equal-spaced arrays. The efficiency-bandwidth product, power-bandwidth product, and power-bandwidth product per unit are all very high in comparison to the previous arrays.

When compared to a two-source equal-level in-phase array, a five source Bessel array is 2.4 dB less efficient, can handle 1.75 (+2.4 dB) more power, has the same maximum midband acoustic output power, and is usable for omnidirectional radiation 10 times higher in frequency! A working distance of twenty times the length of the Bessel array is assumed, with the length of the Bessel array (center-to-center measurement) being four times that of the two-source array.

The very non-linear phase behavior with direction angle and frequency appears to be the single major problem with the Bessel array. Whereas a single point source has true omnidirectional radiation, it does not exhibit any variation of phase with angle or frequency (neglecting transport delay between source and sample point). The Bessel array's off-axis variation of phase with angle and frequency makes it very difficult to use it with any other sources. Computation of group delay vs frequency at an angle of 45 degs indicates an oscillatory movement of acoustic position about ±25% of the arrays length.

Eqs (10) clearly show that the high-frequency limit of the Bessel array increases in direct proportion to the working distance from the array. This is in contrast to the behavior of the two and five-source equal-level equal-polarity equal-spaced arrays, where the performance does not change beyond a fairly close distance measured in terms of the array length (about ten times the array length). This means that the Bessel array is not like a conventional source that exhibits a typical near-field/far-field difference in its behavior. The Bessel array does not have a definite near-field/far-field boundary which defines its behavior.

4.5. Seven-Source Bessel Array with Overall C-C Length of 1.5

As noted in Section 1., the seven-source Bessel array actually has six sources instead of seven, because the middle source has a drive level of zero, and thus does not have to be there physically. The space for the removed source must exist to preserve proper operation of the array, however. The length of the seven-source Bessel array was chosen to be 1.5 units (c-c spacing of outside sources). This specific length was selected because it is the length of the seven-source array when composed of the same size units as the five-source array. The characteristics and calculated parameters for the seven-source Bessel array are shown in Table 6.

The efficiency of the seven-source Bessel array is actually about 11% (0.5 dB) less than the efficiency of a single source that makes up the array. The efficiency is also about 22% less than the five-source Bessel array. With the increased power handling of 4.5 (+6.5 dB), this generates a maximum output of 4 watts (+ 6 dB), which is the same as the maximum output of the two-source equal-level equal-polarity array and the five-source Bessel array.

Because of the additional element required and lower bandwidth, this array's power-bandwidth product per unit is less than half that of the five-source Bessel array. For this reason, very few response curves were generated for the seven-source Bessel array because of its relatively poor characteristics.

4.5.1. Polar Responses

Only one polar response was generated for the seven-source Bessel array. This is shown later in Fig. 30, which compares the polars of all the arrays at a specific frequency and working distance.

4.5.2. Frequency Responses

Only one frequency response was calculated for the seven-source Bessel array and is shown later in Fig. 31, where the response is compared with the other analyzed arrays.

4.5.3. Polar Peak-To-Peak Ripple vs Frequency

Fig. 28 exhibits a plot of the seven-source Bessel array's polar peak-to-peak ripple in dB vs frequency for working distances of 5, 10, 20, 1k, and 100k units. At large distances, the ripple attains a minimum plateau value of about 1.0 dB. Close to the array ($D = 5$), the ripple does not go below about 2 dB. In general, the curves are shifted to the left, as compared to the five-source Bessel, which indicates lower frequencies of operation.

4.5.4. Discussion

As stated in the introduction to this section, the disappointing performance of the seven-source Bessel array as compared to the five-source Bessel array, makes it undesirable for practical use.

4.6. Nine-Source Bessel Array with Overall C-C Length of 2.0

As noted in Section 1., the nine-source Bessel array actually has 7 sources instead of 9, because two of the sources have drive levels of zero, and thus do not have to be in the array. The spaces for the removed sources must exist to preserve proper operation of the array, however. The overall length of the nine-source Bessel array is 2.0 units (c-c spacing of outside sources). This length was chosen because it results from using the same size sources as used in the previous five-source arrays. The characteristics and calculated parameters for the nine-source Bessel array are shown in Table 7.

The efficiency of the nine-source Bessel array is actually about 27% (1.4 dB) <u>less</u> than the efficiency of a single source that makes up the array. The efficiency is also about 36% less than the five-source Bessel array. With the increased power handling of 5.5 (+7.4 dB), this generates a maximum output of 4 watts (+ 6 dB), which is the same as the maximum output of the two-source equal-level equal-polarity array and the five and seven-source Bessel arrays. Also, the maximum upper frequency of the nine-source Bessel array (assuming 4 dB p-p ripple) is less than one-eight that of the five-source Bessel array!

Because of the two additional elements required and drastically lower bandwidth, this array's power-bandwidth product per unit is less than one tenth that of the five-source Bessel array! This very poor performance takes it out of the running for any practical application. For this reason, very few response curves were generated for the nine-source Bessel array.

4.6.1. Polar Responses

Only one polar response was generated for the nine-source Bessel array. This is shown later in Fig. 30, which compares the polars of all the arrays at a specific frequency and working distance.

4.6.2. Frequency Responses

Only one frequency response was calculated for the nine-source Bessel array and is shown later in Fig. 31, where the response is compared with the other analyzed arrays.

4.6.3. Polar Peak-To-Peak Ripple vs Frequency

Fig. 29 shows a plot of the nine-source Bessel array's polar peak-to-peak ripple in dB vs frequency for working distances of 5, 10, 20, 1k, and 100k units. At large distances, the ripple attains a minimum plateau value of about 3.6 dB, which is significantly higher than the previous Bessel arrays. As noted for the seven-source Bessel array, the curves are shifted even more to the left, as compared to the five-source Bessel, which indicates an even lower bandwidth of operation.

4.6.4. Discussion

The performance of the nine-source Bessel array is significantly worse than even the seven-source Bessel array, which was previously judged undesirable for practical use. Its much lower efficiency, requirement of two more sources, and very much lower bandwidth, definitely take it out of the running.

5. ARRAY COMPARATIVE ANALYSIS

A comparative analysis was done on all the analyzed arrays. This included: a master comparison table where all the arrays performance factors are shown, a series of performance ranking tables, a comparative display of polar responses and frequency responses, and a graph showing polar ripple vs frequency for all the arrays.

5.1. Tabular Comparison

Table 8. shows a master tabular comparison of all the analyzed arrays, assuming a working distance of 20 units and a peak-to-peak polar ripple of 4 dB. The last four rows of the table indicate the clear superiority of the five-source Bessel array as compared to the other analyzed arrays. The much higher bandwidth of operation is reflected in the high values of all the bandwidth products.

5.2. Performance Rankings

This section displays rankings for each of the analyzed arrays, for all the major array characteristics.

5.2.1. Efficiency

Table 9. shows the comparative ranking of the analyzed arrays for efficiency. As expected, the five-source equal-level equal-polarity equal-spaced array is at the top of the list. However, its high efficiency is mostly offset by its lower bandwidth of operation. The nine-source Bessel array is at the bottom of the list (27% less efficiency than that of a single source!).

5.2.2. Power Handling

The comparative rankings for input power handling are shown in Table 10. The nine-source Bessel array is at the top of this list. This is fortunate because it also has the lowest efficiency (Table 2). It would make a good heater!

5.2.3. Maximum Acoustic Output Power

Table 11. displays the ranking order for the arrays maximum acoustic output power. The five-source equal-level array is seen to head the list. Even though this array provides high acoustic output power, it's high frequency capabilities are limited. As can be seen, most of the analyzed arrays have maximum outputs of four times a single unit.

5.2.4. Maximum Operating Frequency

Table 12. ranks all the analyzed arrays for maximum operating frequency. Excluding the single source, the five-source Bessel array is seen to head the list with a large two-to-one margin. The widely separated two-source array is a the bottom of the list.

5.2.5. Efficiency-Bandwidth Product

The ranking for efficiency-bandwidth product is shown in Table 13. Again, after excluding the single source, the five-source Bessel array's superiority is clearly shown, with a margin of greater than 2.5 over the second place entry, . The nine-source Bessel array is in next to last place.

5.2.6. Power Bandwidth Product

Likewise shown in Table 14., are the rankings for power-bandwidth product. The five-source Bessel array again heads the list, after excluding the single source. The seven and nine-source Bessel arrays do a bit better in this comparison. The wide-separation two-source array is in last place.

5.2.7. Power Bandwidth Product Per Unit

Table 15. lists the rankings for the power-bandwidth product per unit. This parameter is a good figure of merit for comparing the arrays in that it shows how good the performance is on a per-unit basis. The five-source Bessel array is on again on top, with the exception of the single source. The seven-source Bessel array is in a fairly-strong second place position. The nine-source Bessel array is in next to last place with a power-bandwidth product of about one-twelfth that of the five-source Bessel array.

5.3. Polar Response Comparison

Fig. 30 shows shows a comparison of the polars for all the analyzed arrays. All the polars were run at the same frequency (10 Hz) and working distance (20 units). The peak-to-peak polar ripple is listed on each plot. The superiority of the five-source Bessel (Fig. 30d) is clearly evident.

5.4. Frequency Response Comparison

Fig. 31 displays a comparison of off-axis frequency responses for all the analyzed arrays. The response curves were all run at the same off-axis angle (+45 degs), working distance (20 units), and covered the same frequency range (0.1 to 20 Hz). Again, the five-source Bessel array has the smoothest and most extended response.

5.5. Ripple vs Frequency Comparison

Fig. 32 shows a comparison of the polar peak-to-peak ripple vs frequency for all the analyzed arrays, at a working distance of 20 units. The superiority of the five-source Bessel is again, quite clear.

6. CONCLUSIONS

When compared to the other analyzed arrays, the five-source Bessel line array is the clear winner, considering: 1) polar response, 2) off-axis frequency response, 3) bandwidth of operation, 4) efficiency-bandwidth product, 5) power-bandwidth product, and 6) power-bandwidth product per unit.

Considering maximum frequency of operation for omnidirectional radiation, at a typical working distance of twenty times the length of the array, the Bessel array outperforms a same-length five-source equal-level equal-polarity equal-spaced array by a factor of 28 and a one-quarter-length equal-level equal-polarity equal-spaced array by a factor of 10. Its power-bandwidth product exceeds it nearest competitor, a seven-source Bessel array, by a factor of 2 .

The seven and nine-source Bessel line arrays were found to be effectively unusable due to poor performance, as compared to the five-source Bessel array. Their much lower efficiency, requirement of additional sources, and much lower bandwidth, placed them at a severe performance disadvantage.

The Bessel array's singular main problem is its non-linear phase behavior with direction and frequency. This non-linear behavior makes it difficult to use the array in conjunction with any other source. Crossing it over to a high-frequency device would be difficult and would require a high slope crossover to minimize off-axis lobing effects in the crossover region. The off-axis phase vs frequency response of the Bessel array is non-minimum phase and exhibits an oscillating phase characteristic. The Bessel array's 45 degs off-axis group delay vs frequency performance indicates that its time center ranges over a peak-to-peak shift of greater than 25% of the length of the array, as frequency increases.

The Bessel array does not exhibit normal near-field/far-field behavior. Its performance characteristics and high-frequency response get better and better the farther away you are from the array. This is in sharp contrast to the analyzed two and five-source equal-level equal-polarity equal-spaced arrays, where there was a definite shift from near-field behavior where the characteristics changed strongly with working distance, to far-field behavior, where the characteristics changed very little with distance.

An analysis was not done on the twenty five (5x5) element planar (panel) source. Presumable the strong performance advantages of the five-source Bessel line array would carry over to this configuration.

7. ACKNOWLEDGEMENT

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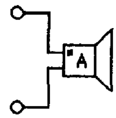
TABLE 1
ARRAY TYPE: SINGLE SOURCE

Configuration:



Number Units (N):	1	
Overall Length (c-c):	0	
Strengths:	1	
Impedance (Z_{in}):	1	(0.0 dB)
Voltage Sensitivity:	1	(0.0 dB)
Efficiency(η_o):	1	(0.0 dB)
Maximum Input Power (P_{in}):	1	(0.0 dB)
Maximum Acoustic Output Power (P_{out}):	1	(0.0 dB)
Maximum Sound Pressure Level:	1	(0.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

Power-Bandwidth Product (P_{out} x F_{max}):

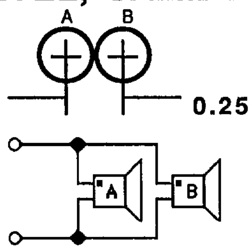
Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5	10	20
Ripple (dB): 3	Infinity	Infinity	Infinity
4	Infinity	Infinity	Infinity
6	Infinity	Infinity	Infinity

TABLE 2
ARRAY TYPE: 2 SOURCES (L=0.25), EQUAL LEVEL, SAME POLARITY

Configuration:



Number Units (N):	2	
Overall Length (c-c):	0.25	
Strengths:	1 : 1	
Impedance (Z_{in}):	0.5	(-3.0 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	2	(+3.0 dB)
Maximum Input Power (P_{in}):	2	(+3.0 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->		5, 10, 20
Ripple (dB):	3	1.00
	4	1.10
	6	1.30

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->		5, 10, 20
Ripple (dB):	3	2.00
	4	2.20
	6	2.60

Power-Bandwidth Product (P_{out} x F_{max}):

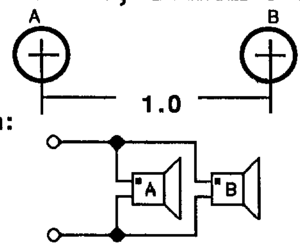
Distance ->		5, 10, 20
Ripple (dB):	3	4.00
	4	4.40
	6	5.20

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->		5, 10, 20
Ripple (dB):	3	2.00
	4	2.20
	6	2.60

TABLE 3
ARRAY TYPE: 2 SOURCES (L=1.0), EQUAL LEVEL, SAME POLARITY

Configuration:



Number Units (N):	2	
Overall Length (c-c):	1.0	
Strengths:	1 : 1	
Impedance (Z_{in}):	0.5	(-3.0 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	2	(+3.0 dB)
Maximum Input Power (P_{in}):	2	(+3.0 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->		5, 10, 20
Ripple (dB):	3	0.25
	4	0.28
	6	0.33

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->		5, 10, 20
Ripple (dB):	3	0.50
	4	0.55
	6	0.65

Power-Bandwidth Product (P_{out} x F_{max}):

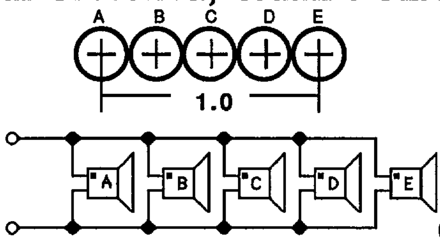
Distance ->		5, 10, 20
Ripple (dB):	3	1.00
	4	1.10
	6	1.30

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->		5, 10, 20
Ripple (dB):	3	0.50
	4	0.55
	6	0.65

TABLE 4
ARRAY TYPE: 5 SOURCES, EQUAL LEVEL, EQUAL SPACING, SAME POLARITY

Configuration:



(Note Polarity Dots!)

Number Units (N):	5	
Overall Length (c-c):	1	
Strengths:	1 : 1 : 1 : 1 : 1	
Impedance (Z_{in}):	1/5 = 0.2	(-7.0 dB)
Voltage Sensitivity:	5	(+14.0 dB)
Efficiency(η_o):	5	(+7.0 dB)
Maximum Input Power (P_{in}):	5	(+7.0 dB)
Maximum Acoustic Output Power (P_{out}):	25	(+14.0 dB)
Maximum Sound Pressure Level:	5	(+14.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->		5, 10, 20
Ripple (dB):	3	0.35
	4	0.40
	6	0.48

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->		5, 10, 20
Ripple (dB):	3	1.75
	4	2.00
	6	2.40

Power-Bandwidth Product (P_{out} x F_{max}):

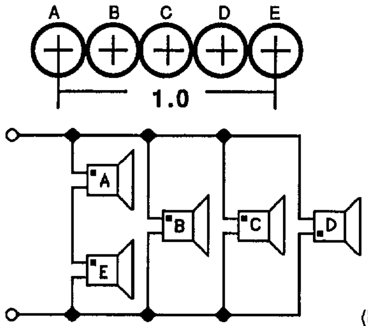
Distance ->		5, 10, 20
Ripple (dB):	3	8.75
	4	10.00
	6	12.00

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->		5, 10, 20
Ripple (dB):	3	1.75
	4	2.00
	6	2.40

TABLE 5
ARRAY TYPE: 5 SOURCE BESSEL

Configuration:



(Note Polarity Dots!)

Number Units (N):	5	
Overall Length (c-c):	1.0	
Strengths:	0.5 : 1 : 1 : -1 : 0.5	
Impedance (Z_{in}):	2/7 = 0.286	(-5.4 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	8/7 = 1.143	(+0.6 dB)
Maximum Input Power (P_{in}):	7/2 = 3.5	(+5.4 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	2.05	4.00	8.00
4	3.00	6.00	11.00
6	4.50	8.80	18.00

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	2.34	4.57	9.14
4	3.43	6.86	12.57
6	5.14	10.06	20.57

Power-Bandwidth Product (P_{out} x F_{max}):

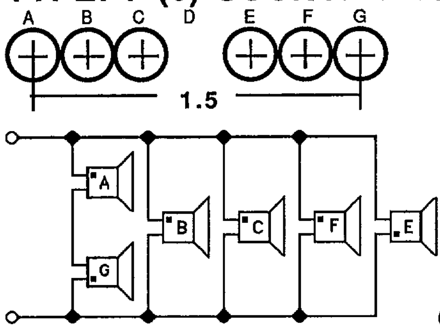
Distance ->	5	10	20
Ripple (dB): 3	8.20	16.00	32.00
4	12.00	24.00	44.00
6	18.00	35.20	72.00

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5	10	20
Ripple (dB): 3	1.64	3.20	6.40
4	2.40	4.80	8.80
6	3.60	7.04	14.40

TABLE 6
ARRAY TYPE: 7 (6) SOURCE BESSEL

Configuration:



(Note Polarity Dots!)

Number Units (N):	6		
Overall Length (c-c):	1.5		
Strengths:	0.5 : 1 : 1 : 0 : -1 : 1 : -0.5		
Impedance (Z_{in}):	2/9 = 0.222	(-6.5 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	8/9 = 0.889	(-0.5 dB)
Maximum Input Power (P_{in}):	9/2 = 4.5	(+6.5 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	1.36	2.20	4.00
4	1.73	2.95	5.55
6	2.43	4.32	8.30

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	1.21	1.96	3.56
4	1.54	2.62	4.93
6	2.16	3.84	7.38

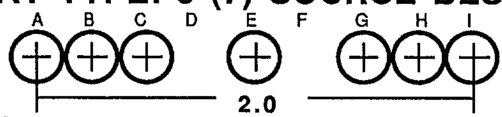
Power-Bandwidth Product (P_{out} x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	5.44	8.80	16.00
4	6.92	11.80	22.20
6	9.72	17.28	33.20

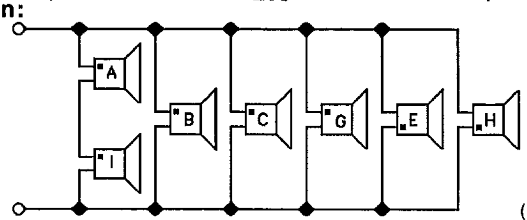
Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5	10	20
Ripple (dB): 3	0.91	1.47	2.67
4	1.15	1.97	3.70
6	1.62	2.88	5.53

TABLE 7
ARRAY TYPE: 9 (7) SOURCE BESSEL



Configuration:



(Note Polarity Dots!)

Number Units (N):	7	
Overall Length (c-c):	2.0	
Strengths:	0.5 : 1 : 1 : 0 : -1 : 0 : 1 : -1 : 0.5	
Impedance (Z_{in}):	2/11 = 0.182	(-7.4 dB)
Voltage Sensitivity:	2	(+6.0 dB)
Efficiency(η_o):	8/11 = 0.727	(-1.4 dB)
Maximum Input Power (P_{in}):	11/2 = 5.5	(+7.4 dB)
Maximum Acoustic Output Power (P_{out}):	4	(+6.0 dB)
Maximum Sound Pressure Level:	2	(+6.0 dB)

Maximum Upper Frequency (F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	0.13	0.46	0.78
4	0.46	1.06	1.30
6	1.50	1.96	3.95

Efficiency-Bandwidth Product (η_o x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	0.09	0.33	0.56
4	0.33	0.77	0.95
6	1.09	1.42	2.87

Power-Bandwidth Product (P_{out} x F_{max}):

Distance ->	5	10	20
Ripple (dB): 3	0.52	1.84	3.12
4	1.84	4.24	5.20
6	6.00	7.84	15.80

Power-Bandwidth Product per Unit (P_{out} x F_{max} / N):

Distance ->	5	10	20
Ripple (dB): 3	0.07	0.26	0.44
4	0.26	0.61	0.74
6	0.86	1.12	2.26

TABLE 8. COMPARISON OF ARRAY TYPES

ARRAY TYPE =	Single Source	2 Sources Equal Level (L=0.25)	2 Sources Equal Level (L=1.0)	5 Sources Equal Level & Spacing	5 Source Bessel	7(6) Source Bessel	9(7) Source Bessel
Configuration (to scale) =	o	oo	o o	ooooo	ooooo	ooo ooo	ooo o ooo
Number Units =	1	2	2	5	5	6	7
Overall Length (c-c) =	0	0.25	1.0	1.0	1.0	1.5	2.0
Impedance =	1.000	0.500	0.500	0.200	0.286	0.222	0.182
Voltage Sensitivity =	1	2	2	5	2	2	2
Efficiency =	1.000	2.000	2.000	5.000	1.143	0.889	0.727
Maximum Input Power =	1.0	2.0	2.0	5.0	3.5	4.5	5.5
Max. Output Power =	1.0	4.0	4.0	25.0	4.0	4.0	4.0
Maximum Sound Pressure Level =	1.0 (0 dB)	2.0 (+6 dB)	2.0 (+6 dB)	5.0 (+14 dB)	2.0 (+6 dB)	2.0 (+6 dB)	2.0 (+6 dB)
Maximum Upper Frequency = (Distance = 20, P-P Ripple = 4 dB)	Infinity	1.10	0.28	0.40	11.00	5.55	1.30
Efficiency-Bandwidth Product = (Distance = 20, P-P Ripple = 4 dB)	Infinity	2.20	0.56	2.00	12.57	4.93	0.95
Power Bandwidth Product = (Distance = 20, P-P Ripple = 4 dB)	Infinity	4.4	1.1	10.0	44.0	22.2	5.2
Power-Bandwidth Product per Unit = (Distance = 20, P-P Ripple = 4 dB)	Infinity	2.20	0.56	2.00	8.80	3.70	0.74

TABLE 9
RANKING FOR EFFICIENCY

RANK	VALUE	ARRAY TYPE
1	5.00	5 Sources, Equal Level and Spacing
2	2.00	2 Sources, Equal Level (L=0.25)
3	2.00	2 Sources, Equal Level (L=1.0)
4	1.14	5 Source Bessel
5	1.00	Single Source
6	0.89	7(6) Source Bessel
7	0.73	9(7) Source Bessel

TABLE 10
RANKING FOR MAXIMUM INPUT POWER

RANK	VALUE	ARRAY TYPE
1	5.5	9(7) Source Bessel
2	5.0	5 Sources, Equal Level and Spacing
3	4.5	7(6) Source Bessel
4	3.5	5 Source Bessel
5	2.0	2 Sources, Equal Level (L=0.25)
6	2.0	2 Sources, Equal Level (L=1.0)
7	1.0	Single Source

TABLE 11
RANKING FOR MAXIMUM OUTPUT POWER

RANK	VALUE	ARRAY TYPE
1	25	5 Sources, Equal Level and Spacing
2	4	2 Sources, Equal Level (L=0.25)
3	4	2 Sources, Equal Level (L=1.0)
4	4	5 Source Bessel
5	4	7(6) Source Bessel
6	4	9(7) Source Bessel
7	1	Single Source

TABLE 12
RANKING FOR MAXIMUM OPERATING FREQUENCY

(Distance = 20, P-P Ripple = 4 dB)

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	11.00	5 Source Bessel
3	5.55	7(6) Source Bessel
4	1.30	9(7) Source Bessel
5	1.10	2 Sources, Equal Level (L=0.25)
6	0.40	5 Sources, Equal Level and Spacing
7	0.28	2 Sources, Equal Level (L=1.0)

TABLE 13
RANKING FOR EFFICIENCY-BANDWIDTH PRODUCT

(Distance = 20, P-P Ripple = 4 dB)

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	12.57	5 Source Bessel
3	4.93	7(6) Source Bessel
4	2.20	2 Sources, Equal Level (L=0.25)
5	2.00	5 Sources, Equal Level and Spacing
6	0.95	9(7) Source Bessel
7	0.56	2 Sources, Equal Level (L=1.0)

TABLE 14
RANKING FOR POWER-BANDWIDTH PRODUCT

(Distance = 20, P-P Ripple = 4 dB)

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	44.0	5 Source Bessel
3	22.2	7(6) Source Bessel
4	10.0	5 Sources, Equal Level and Spacing
5	5.2	9(7) Source Bessel
6	4.4	2 Sources, Equal Level (L=0.25)
7	1.1	2 Sources, Equal Level (L=1.0)

TABLE 15
RANKING FOR POWER-BANDWIDTH PRODUCT per UNIT

(Distance = 20, P-P Ripple = 4 dB)

RANK	VALUE	ARRAY TYPE
1	Infinity	Single Source
2	8.80	5 Source Bessel
3	3.70	7(6) Source Bessel
4	2.20	2 Sources, Equal Level (L=0.25)
5	2.00	5 Sources, Equal Level and Spacing
6	0.74	9(7) Source Bessel
7	0.56	2 Sources, Equal Level (L=1.0)

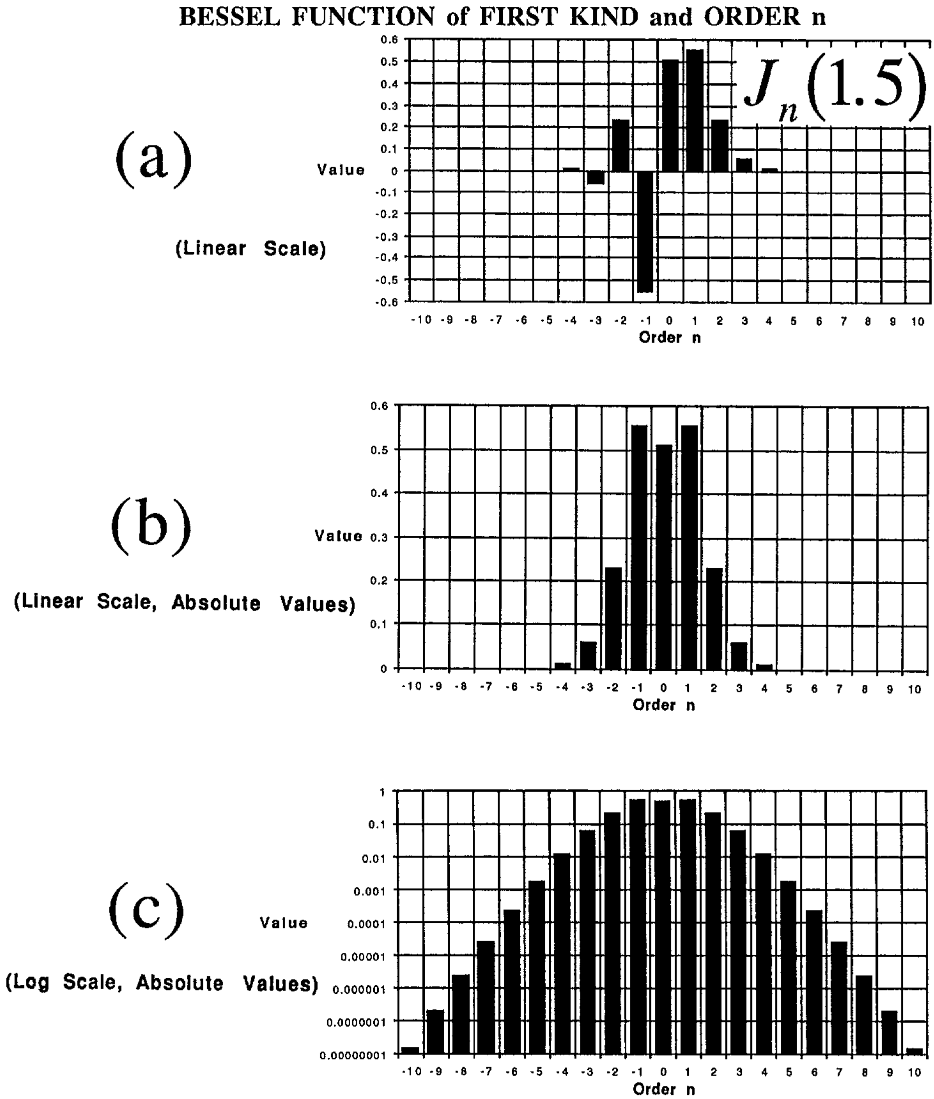


Fig. 1. Values of the Bessel function of first kind and order n for an argument of 1.5 ($J_n(1.5)$), over the range -10 <= n <= +10, plotted in bar graph form. (a) Plot using a linear vertical scale. (b) Plot of absolute values using a linear vertical scale. (c) Plot of absolute values using a logarithmic vertical scale covering the range of 10^{-8} to 1.0. Note that the values rapidly grow very small for n beyond ±3. The values in the range of $|n|$ <= 2 are used to generate the source drive levels and polarities for the five-source Bessel array (+0.5 : -1 : +1 : +1 : +0.5).

TWO-SOURCE ARRAY, 0.25 UNIT C-C SPACING
Polars at Constant Distance, Vary Frequency

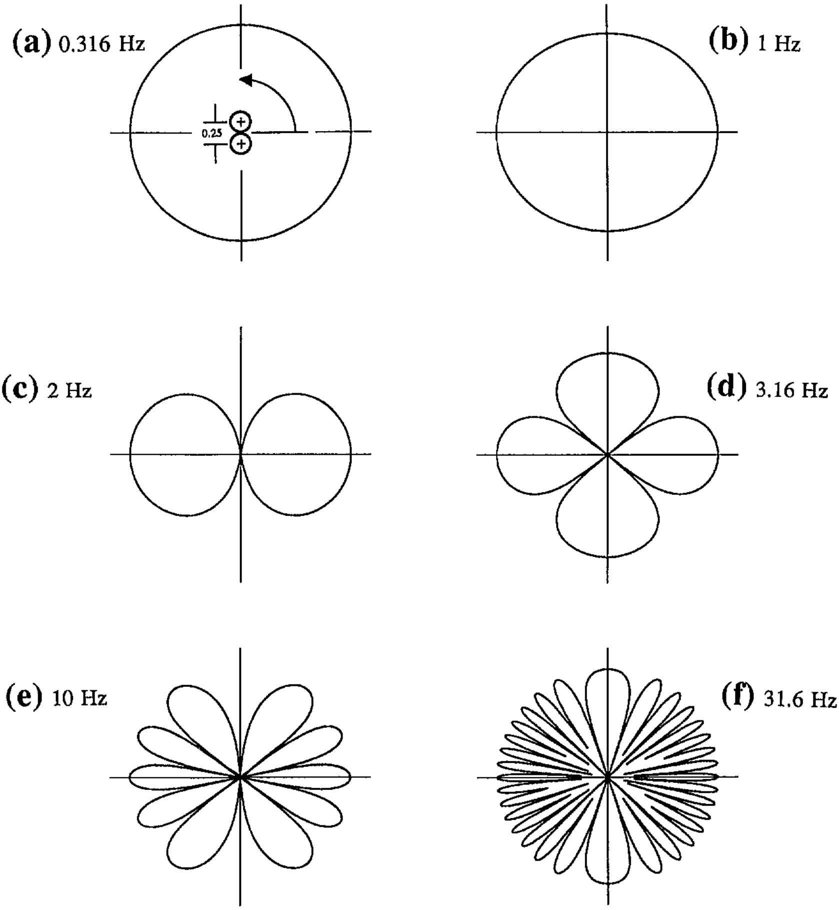


Fig. 2. Polar magnitude responses for the two-source equal-level equal-polarity array with 0.25 unit c-c spacing, at a constant working distance of 20 units. This spacing is the same as the individual c-c spacing of the five-source arrays. The polars are displayed at half-decade intervals from 0.316 Hz to 31.6 Hz with an additional polar at 2.0 Hz. (a) 0.316 Hz. (b) 1 Hz. (c) 2 Hz. (d) 3.16 Hz. (e) 10 Hz. (f) 31.6 Hz. The polar plot covers a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB. The sources are one-half wavelength apart at 2 Hz and exhibit a null at ±90 degs off axis (c). Note that polar response is mostly omnidirectional at and below 1 Hz, but gets progressively narrower and gains additional lobes as frequency increases.

TWO-SOURCE ARRAY, 0.25 UNIT C-C SPACING
Polars at Constant Frequency, Vary Distance

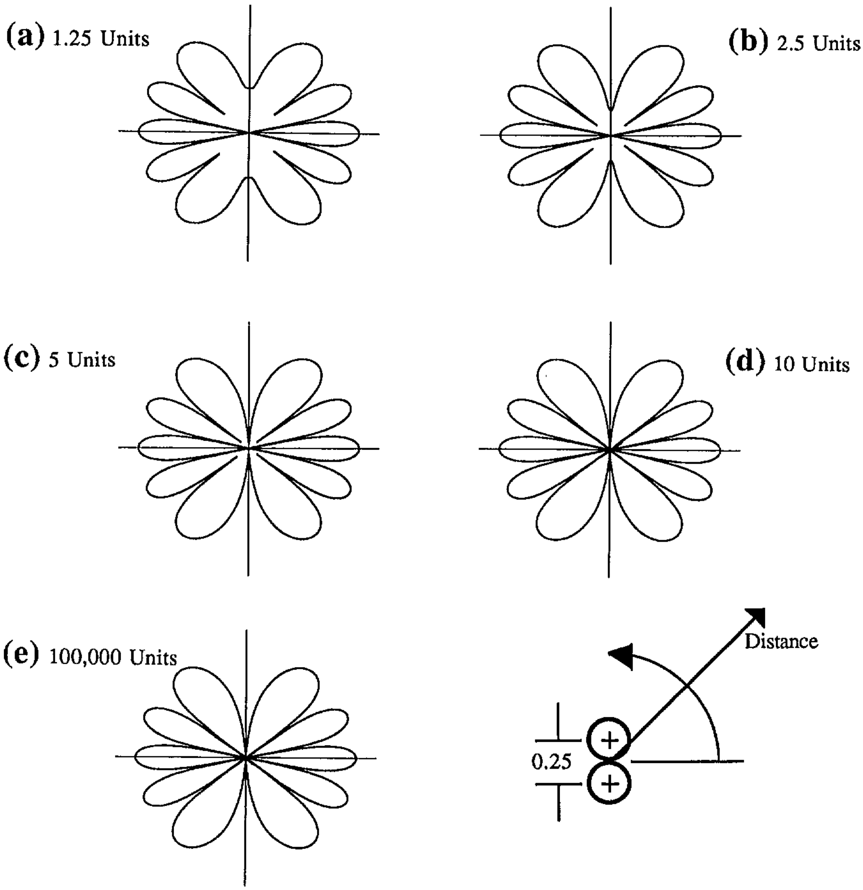


Fig. 3. Polar magnitude responses for the two-source equal-level equal-polarity array with 0.25 unit c-c spacing, at a fixed frequency of 10 Hz and at different working distances. Polars at distances of (a) 1.25, (b) 2.5, (c) 5, (d) 10 and (e) 100k units are shown. Observe that the polar responses essentially exhibit no change with increasing working distance beyond about 2.5 units (10 times array length).

TWO-SOURCE ARRAY, 1.0 UNIT C-C SPACING
Polars and Phase vs Direction Angle

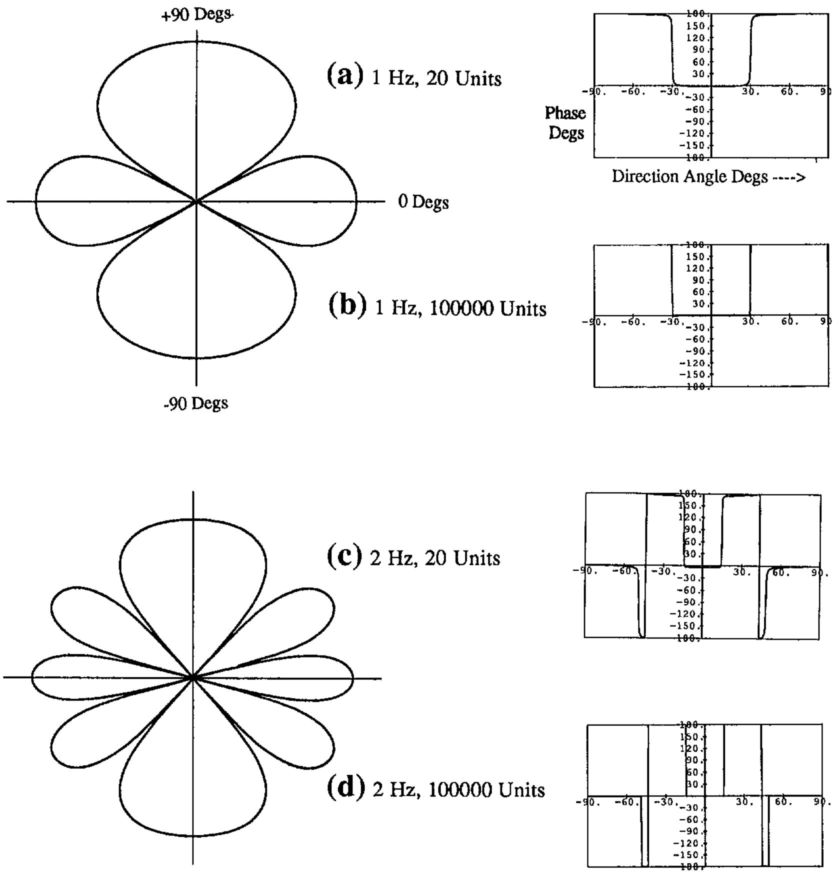


Fig. 4. Polar magnitude and phase responses, at various frequencies and distances, for the two-source equal-level equal-polarity array with 1.0 c-c spacing. The phase vs direction plots show the phase of the pressure at the sample point vs the off-axis direction. The following four combinations are plotted: (a) $f = 1$ Hz, $D = 20$ units; (b) $f = 1$ Hz, $D = 100$k; (c) $f = 2$ Hz, $D = 20$; and (d) $f = 2$ Hz, $D = 100$k. The phase values are referenced to the input signal of the array. The effects of linear phase lag and delay due to sample distance have been removed. Note how the phase changes rapidly from 0 to ±180 degs as direction angle increases, as each separate lobe is transversed. Note also, that this array is spaced one-half wavelength apart at 0.5 Hz.

TWO-SOURCE ARRAY, 1.0 UNIT SPACING
Frequency Response vs Direction Angle, Constant Distance

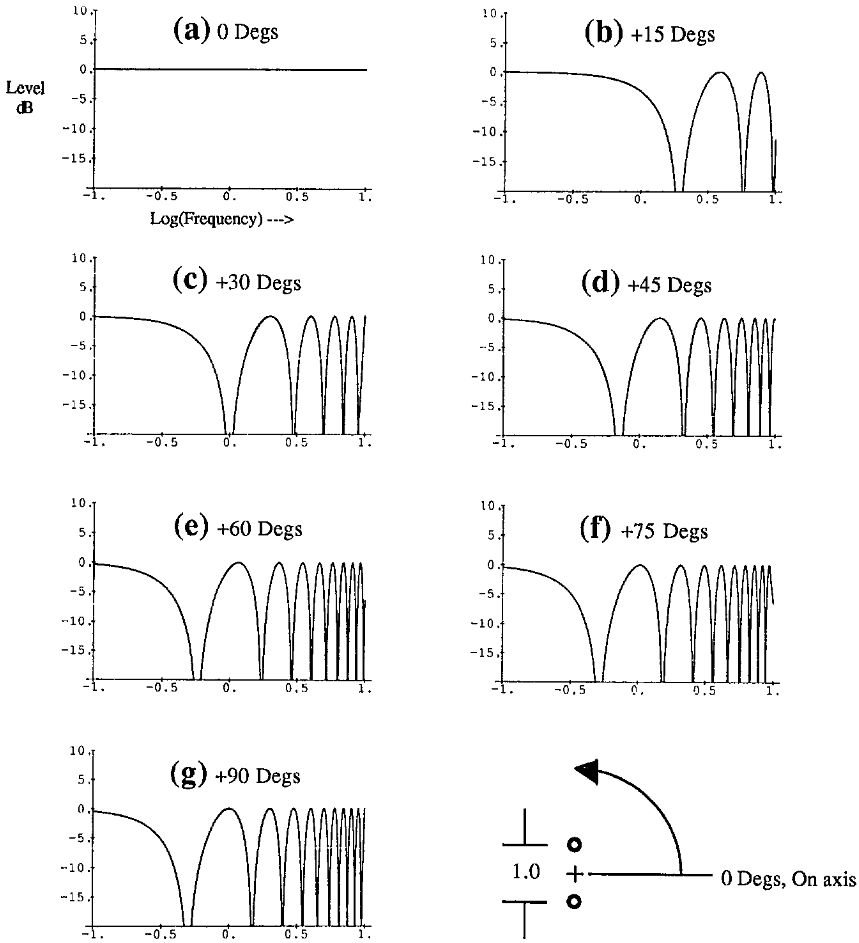


Fig. 5. Magnitude frequency responses of the two-source equal-level equal-polarity array with 1.0 c-c spacing, at constant distance of 20 units, with frequency range of 0.1 to 10 Hz. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.). The responses are shown at angles ranging from 0 to +90 degs with steps of 15 degs. (a) 0 degs. (b) 15 degs. (c) 30 degs. (d) 45 degs. (e) 60 degs. (f) 75 degs. (g) 90 degs. Note that the response progressively gets rougher as the angle increases.

TWO-SOURCE ARRAY, 1.0 UNIT SPACING
Frequency Response vs Distance, Constant Angle

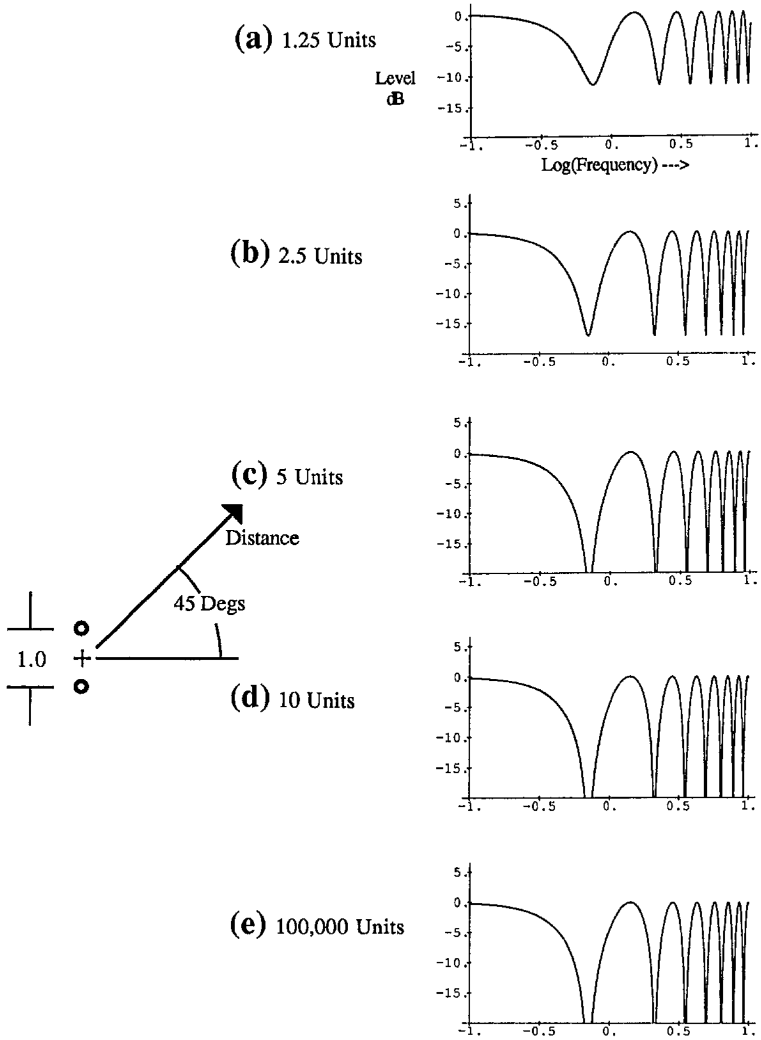


Fig. 6. Magnitude frequency responses, at a fixed angle of 45 degs and at different working distances, for the two-source equal-level equal-polarity array with 1.0 c-c spacing, with frequency range of 0.1 to 10 Hz. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.). Distances of (a) 1.25, (b) 2.5, (c) 5, (d) 10, and (e) 100k units are shown. Observe that the frequency response changes very little with distance beyond 5 units.

TWO-SOURCE ARRAY, 1.0 UNIT SPACING
Magnitude and Phase vs Frequency, Constant Angle

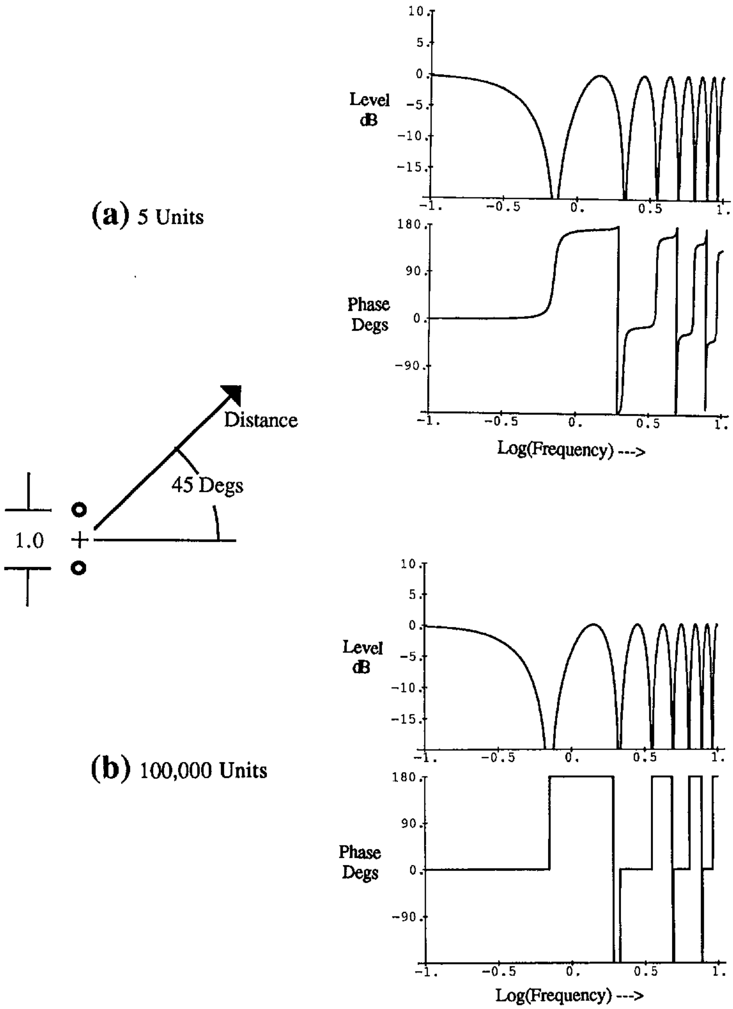


Fig. 7. Off-axis, +45 degs, magnitude and phase vs frequency responses for the two-source equal-level equal-polarity array with 1.0 c-c spacing, at distances of (a) 5 and (b) 100k units. Observe that the phase is either 0 or ±180 degrees, depending on which polar lobe the sample point happens to be in. At the farther distance, the phase switches very rapidly. The phase vs frequency behavior is near non-minimum phase but is actually minimum-phase.

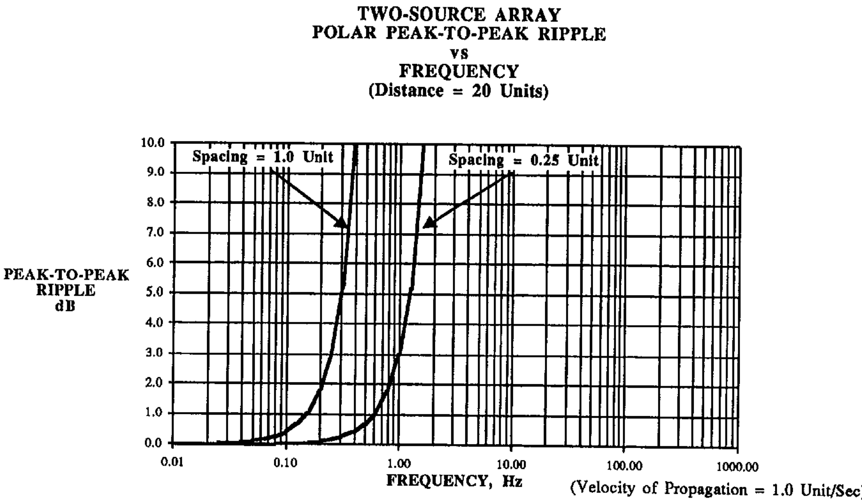


Fig. 8. Polar peak-to-peak ripple in dB vs frequency for both two-source equal-level equal-polarity arrays at a working distance of 20 units. Note that the ripple increases very rapidly above 1 Hz for the 0.25 unit spaced array and above 0.25 Hz for the 1.0 unit spaced array.

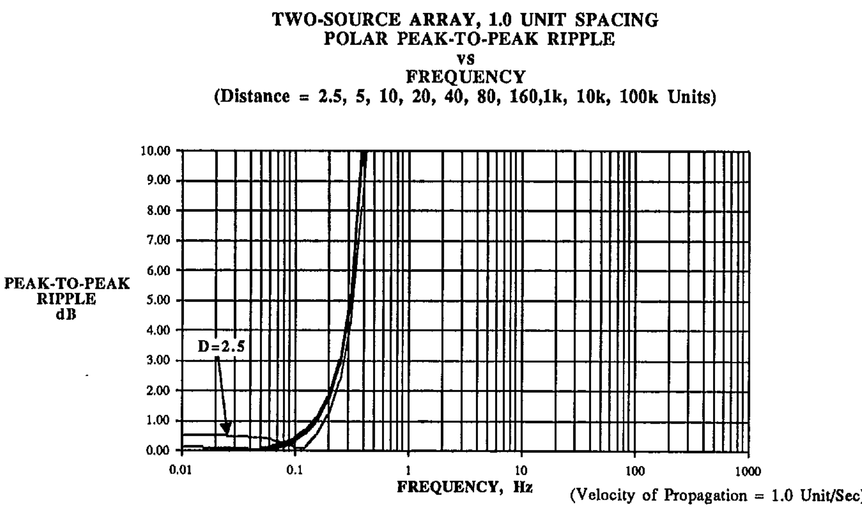


Fig. 9. Polar peak-to-peak ripple in dB vs frequency at working distances of 2.5, 5, 10, 20, 40, 80, 160, 1k, 10k and 100k units for the 1.0 spaced two-source equal-level equal-polarity array. The graph exhibits essentially no change at distances beyond 5 units. Note the close bunching of all the curves.

TWO-SOURCE ARRAY, 0.25 UNIT SPACING
Polar Response at 1.1 Hz with 4 dB Peak-to-Peak Ripple

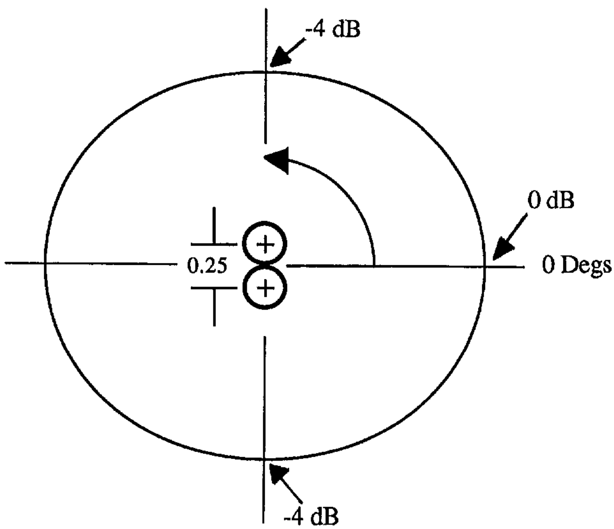


Fig. 10.Magnitude polar response of the 0.25 c-c spaced two-source equal-level equal-polarity array, at a distance of 20 units and a frequency of 1.1 Hz. This polar is at the frequency where the peak-to-peak ripple is 4 dB. Note that the polar is very smooth, but squashed vertically, and exhibits its maximum deviation (-4 dB) at ±90 degs. The polar plot covers a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. The polar is normalized so that the on-axis level is 0 dB.

FIVE-SOURCE EQUAL-LEVEL ARRAY,
1.0 UNIT C-C LENGTH
Polars at Constant Distance, Vary Frequency

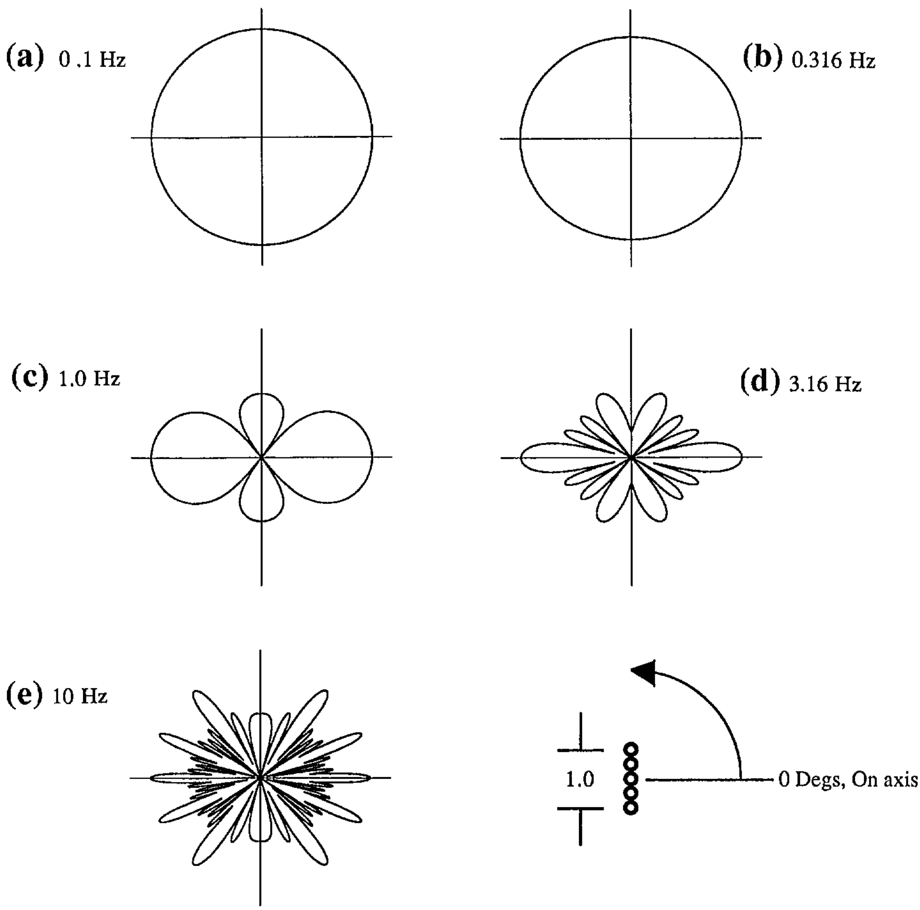


Fig. 11. Polar magnitude responses of the five-source unit-length equal-level equal-polarity equal-spaced array, at a constant working distance of 20 units. The polars are displayed at half-decade intervals from 0.1 Hz to 10 Hz. (a) 0.1 Hz. (b) 0.316 Hz. (c) 1 Hz. (d) 3.16 Hz. (e) 10 Hz. Note how directive and complex the polars get above 0.316 Hz. The polar plot covers a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB.

FIVE-SOURCE EQUAL-LEVEL ARRAY,
1.0 UNIT C-C LENGTH
Polars at Constant Frequency, Vary Distance

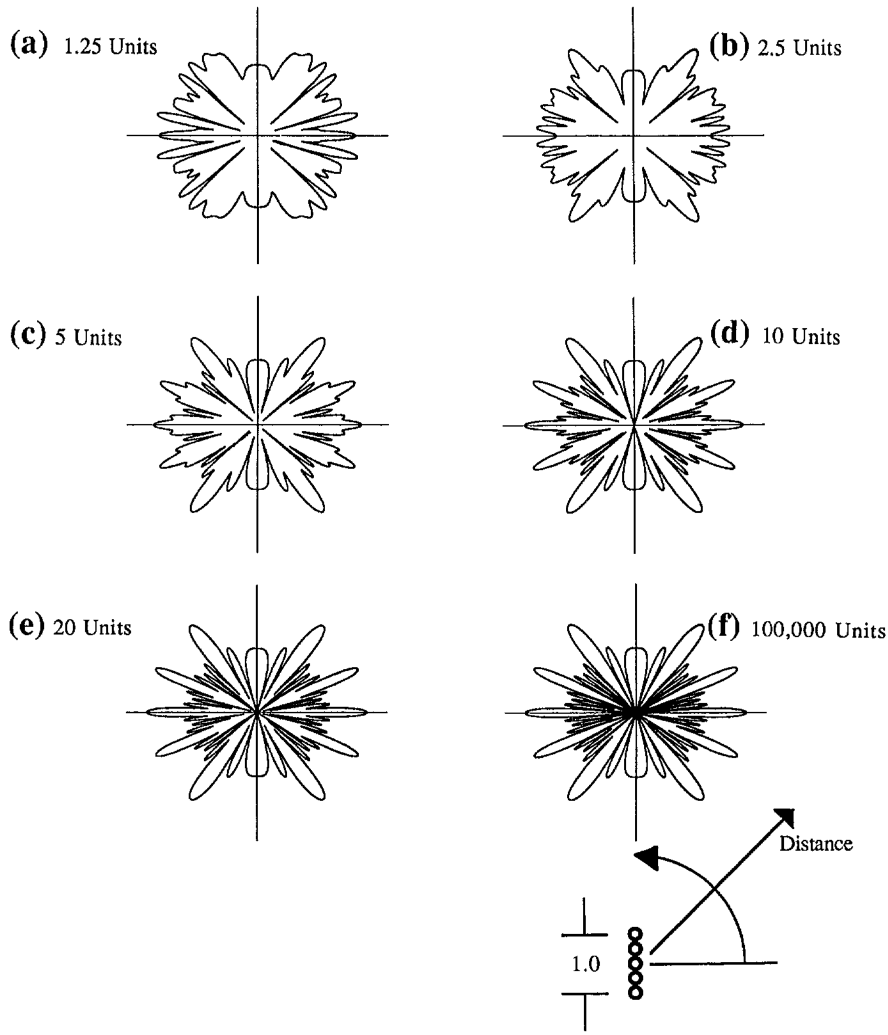


Fig. 12. Polar magnitude responses for the five-source unit-length equal-level equal-polarity equal-spaced array, at a fixed frequency of 10 Hz and at different working distances. Distances of: (a) 1.25, (b) 2.5, (c) 5, (d) 10, (e) 20 and (f) 100k units are shown. Note that the polar response changes very little with distance beyond roughly 10 units (10 array lengths).

FIVE-SOURCE EQUAL-LEVEL ARRAY, 1.0 UNIT C-C LENGTH
Polars and Phase vs Direction Angle

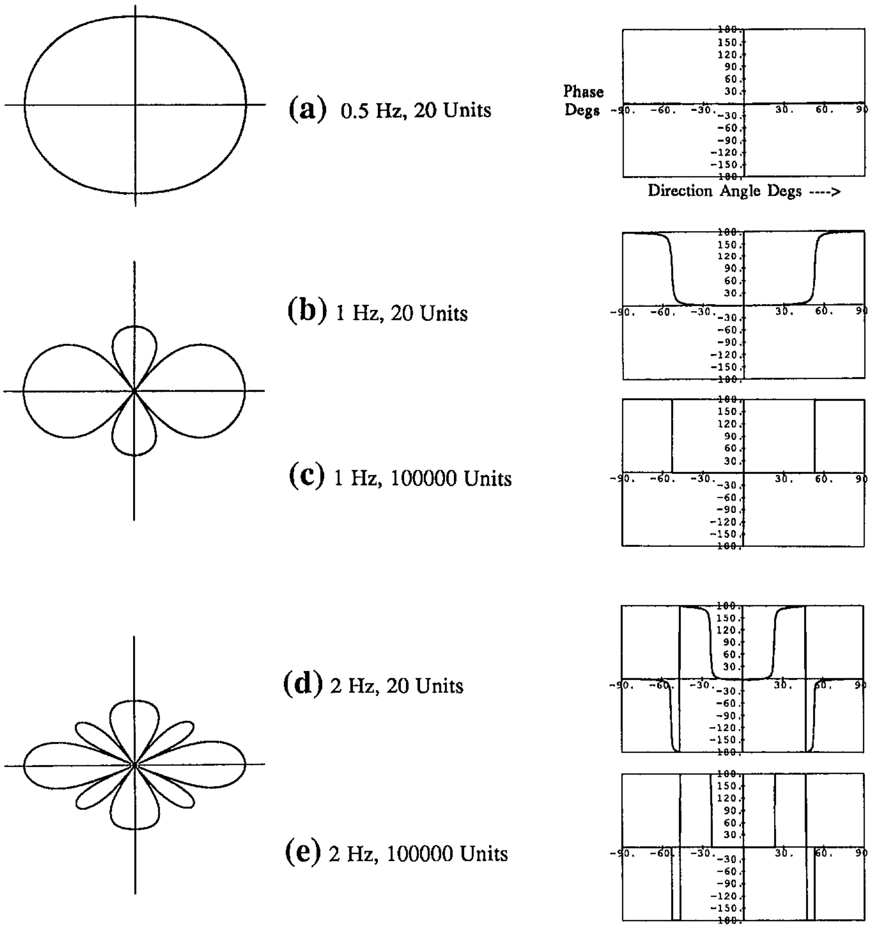


Fig. 13. Magnitude and phase polar responses at different frequencies and distances for the five-source unit-length equal-level equal-polarity equal-spaced array. The following five combinations are plotted: (a) $f = 0.5$ Hz, $D = 20$ units; (b) $f = 1$ Hz, $D = 20$; (c) $f = 1$ Hz, $D = 100$k; (d) $f = 2$ Hz, $D = 20$; and (e) $f = 2$ Hz, $D = 100$k. The phase vs direction plots show the phase of the pressure at the sample point vs the off-axis direction. The effects of linear phase delay due to sample distance have been eliminated. Note that the phase values switch between 0 and ±180 degrees depending on which polar lobe the pressure sample point happens to be on. The phase always starts out at zero degrees (on axis). At distances far from the array, the phase changes occur more abruptly with angle.

FIVE-SOURCE EQUAL-LEVEL ARRAY,
1.0 UNIT C-C LENGTH
Frequency Response vs Direction Angle, Constant Distance

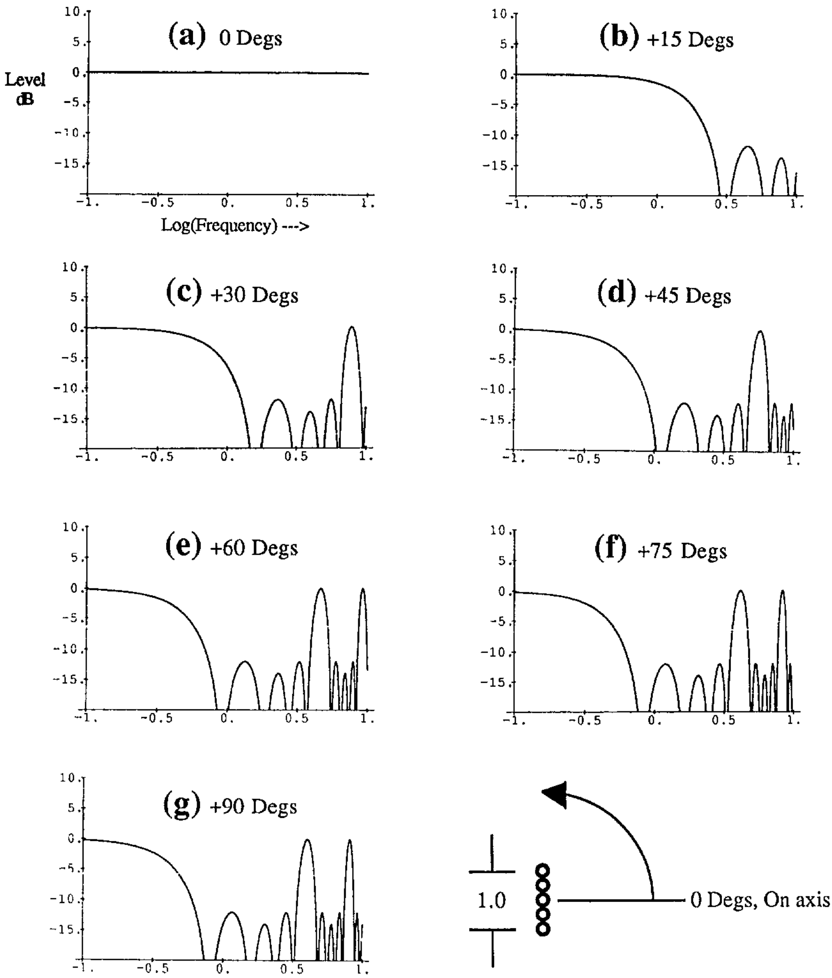


Fig. 14. Magnitude vs frequency responses for the five-source unit-length equal-level equal-polarity equal-spaced array, at a constant distance of 20 units, with frequency range of 0.1 to 10 Hz. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.). The responses are shown at angles ranging from 0 to +90 degs with steps of 15 degs. (a) 0 degs. (b) 15 degs. (c) 30 degs. (d) 45 degs. (e) 60 degs. (f) 75 degs. (g) 90 degs. Note that the response progressively gets rougher as the angle increases, similarly to the two-source arrays.

FIVE-SOURCE EQUAL-LEVEL ARRAY,
1.0 UNIT C-C LENGTH
Frequency Response vs Distance, Constant Angle

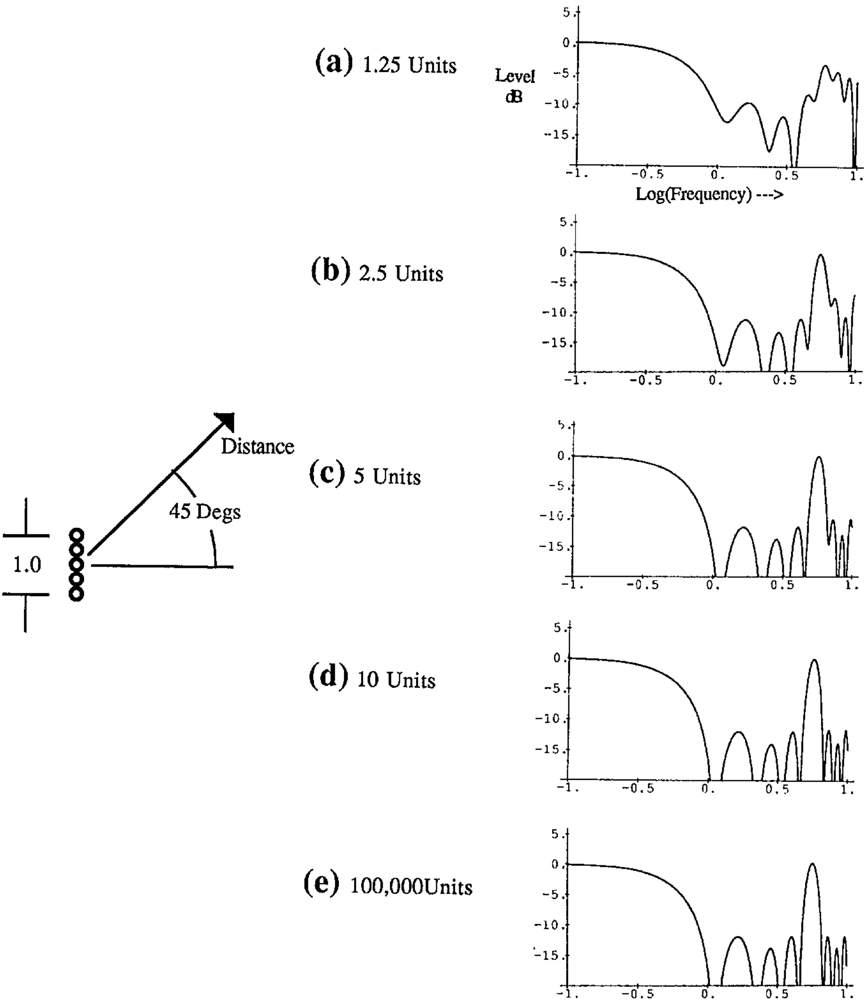


Fig. 15. Magnitude vs frequency responses, at a fixed angle of 45 degs and at different working distances, for the five-source unit-length equal-level equal-polarity equal-spaced array, with frequency range of 0.1 to 10 Hz. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.). Distances of: (a) 1.25, (b) 2.5, (c) 5, (d) 10, and (e) 100k units are shown. Note that the frequency response changes very little with distance beyond about 5 units.

FIVE-SOURCE EQUAL-LEVEL ARRAY,
1.0 UNIT C-C LENGTH
Magnitude and Phase vs Frequency, Constant Angle

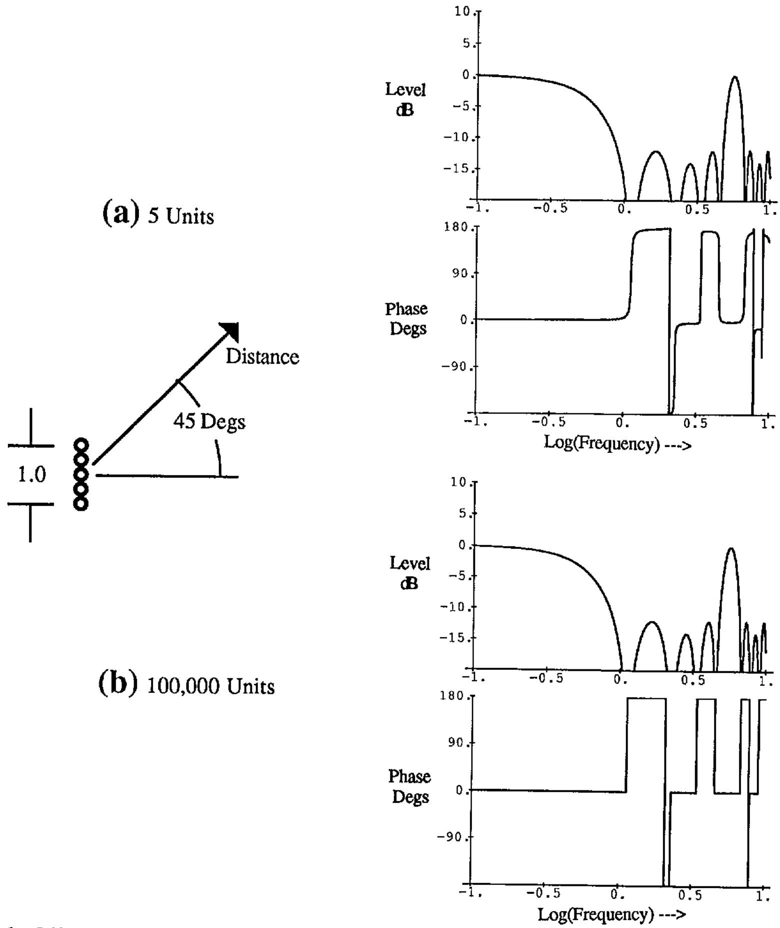


Fig. 16. Off-axis, 45 degs, magnitude and phase vs frequency plots, over the frequency range of 0.1 to 10 Hz, for the five-source unit-length equal-level equal-polarity equal-spaced array at distances of (a) 20 and (b) 100k units. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.). The effects of linear phase lag and delay due to sample distance have been eliminated. The phase activity vs frequency is very similar to the two-source arrays, but is highly likely to be non-minimum phase due to the existence of the additional sources. The phase toggles rapidly between 0 and ±180 degrees as frequency increases.

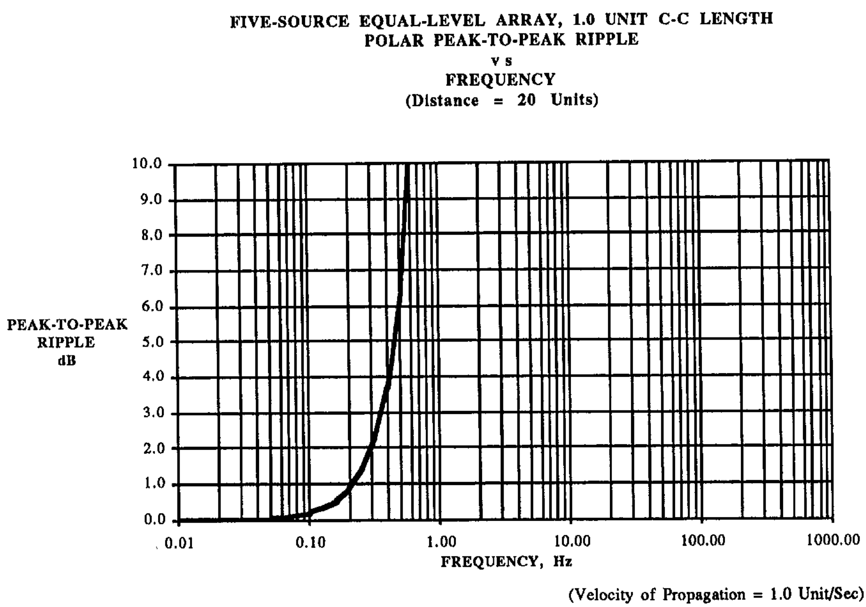


Fig. 17. Plot of polar peak-to-peak ripple in dB vs frequency for the five-source unit-length equal-level equal-polarity equal-spaced array, at a working distance of 20 units. Note that the ripple increases very rapidly above 0.35 Hz. The five-source unit-length array has somewhat better performance than the two-source unit-length array but significantly lower than the 0.25 unit two-source array (see Fig. 8).

FIVE-SOURCE BESSEL ARRAY, 1.0 UNIT C-C LENGTH
Polars at Constant Distance, Vary Frequency

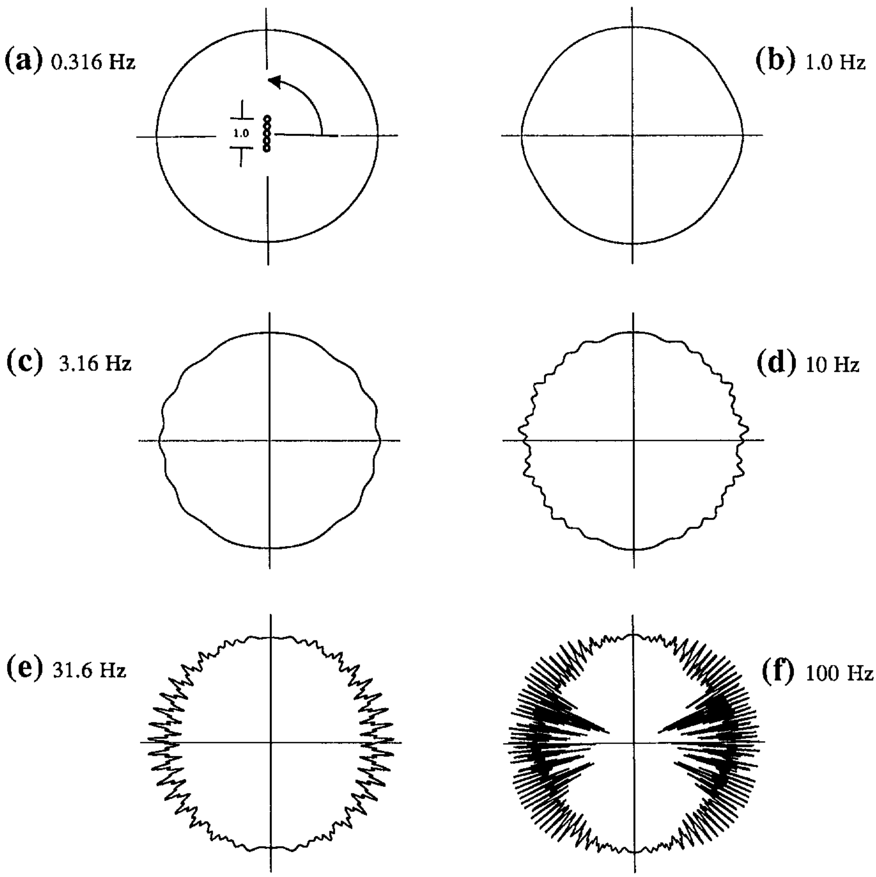


Fig 18. Polar magnitude responses for the five-source unit-length Bessel array, at a constant working distance of 20 units. The polars are displayed at half-decade intervals from 0.316 Hz to 100 Hz. (a) 0.316 Hz. (b) 1 Hz. (c) 3.16 Hz. (d) 10 Hz. (e) 31.6 Hz. (f) 100 Hz. Note the much greater high frequency range of operation as compared with the previous arrays. The polar ripple does not get significant until frequencies higher than about 10 Hz where the line length is 10 wavelengths. The polar plots cover a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB.

FIVE-SOURCE BESSEL ARRAY,
1.0 UNIT C-C LENGTH
Polars at Constant Frequency, Vary Distance

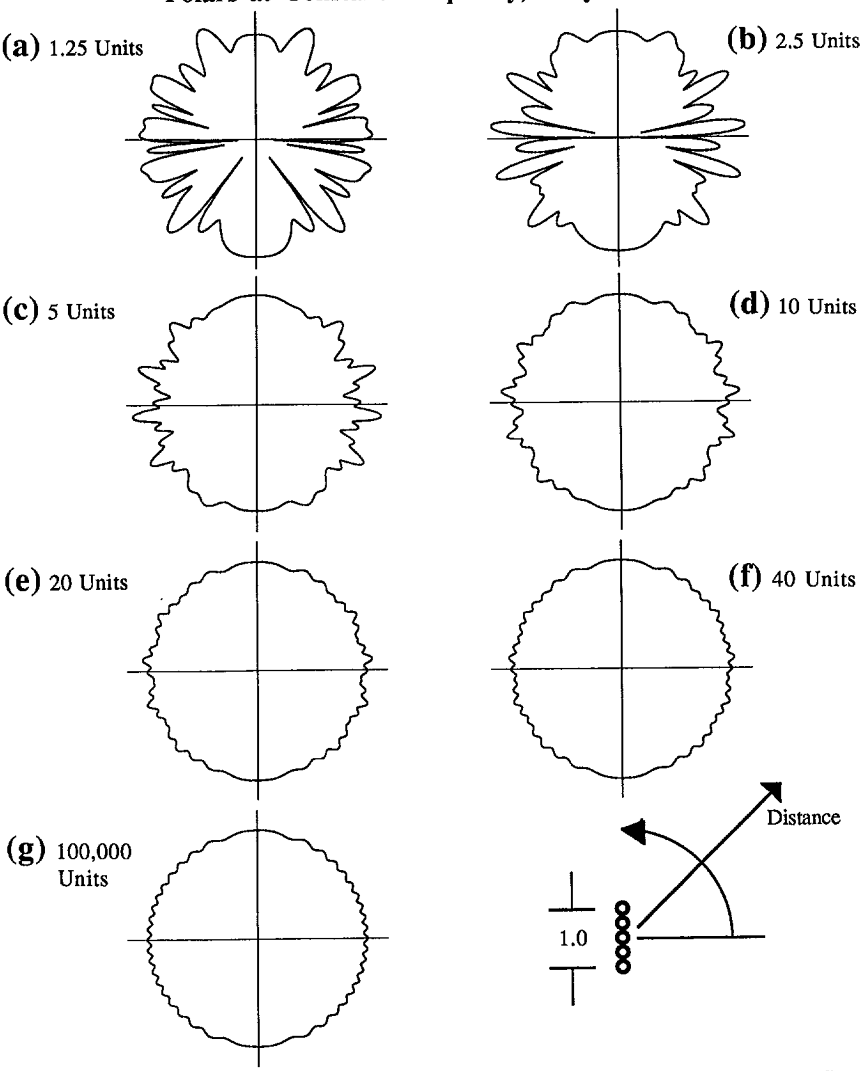
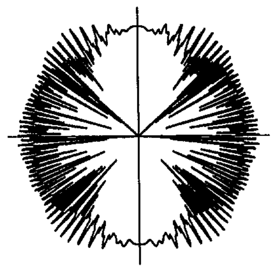


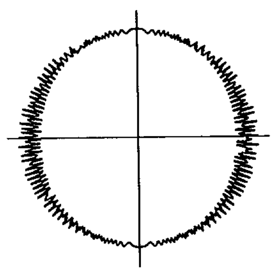
Fig.19. Polar magnitude responses for the five-source unit-length Bessel array, at a fixed frequency of 10 Hz and at different working distances. Polars at distances of (a) 1.25, (b) 2.5, (c) 5, (d) 10, (e) 20, (f) 40, and (g) 100k units are shown. Note that, unlike the previous arrays, the polar ripple appears to get smaller and smaller the farther away you get from the array! The polar plots cover a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB.

FIVE-SOURCE BESSEL ARRAY,
1.0 UNIT C-C LENGTH
Polars at a Constant *High* Frequency, Vary Distance

(a) 10 Units



(b) 100 Units



(c) 1,000 Units

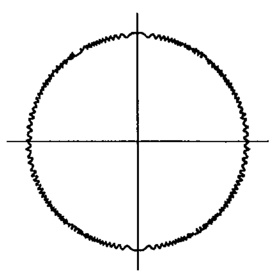


Fig. 20. Polar magnitude responses for the five-source unit-length Bessel array, but at a much higher fixed frequency of 100 Hz and at much farther working distances covering the range of 10 to 1000 units in three decade steps. Polars at distances of (a) 10, (b) 100, and (c) 1000 units are shown. Note that even at this high frequency, where the line is 100 wavelengths long, that at large distances the polar p-p ripple settles down to relatively small values! The polar plots cover a range of 40 dB with +6 dB at the outer edge and -34 dB at the center. All polars are normalized so that the on-axis level is 0 dB.

FIVE-SOURCE BESSEL ARRAY, 1.0 UNIT C-C LENGTH
Phase vs Direction Angle, Constant Distance

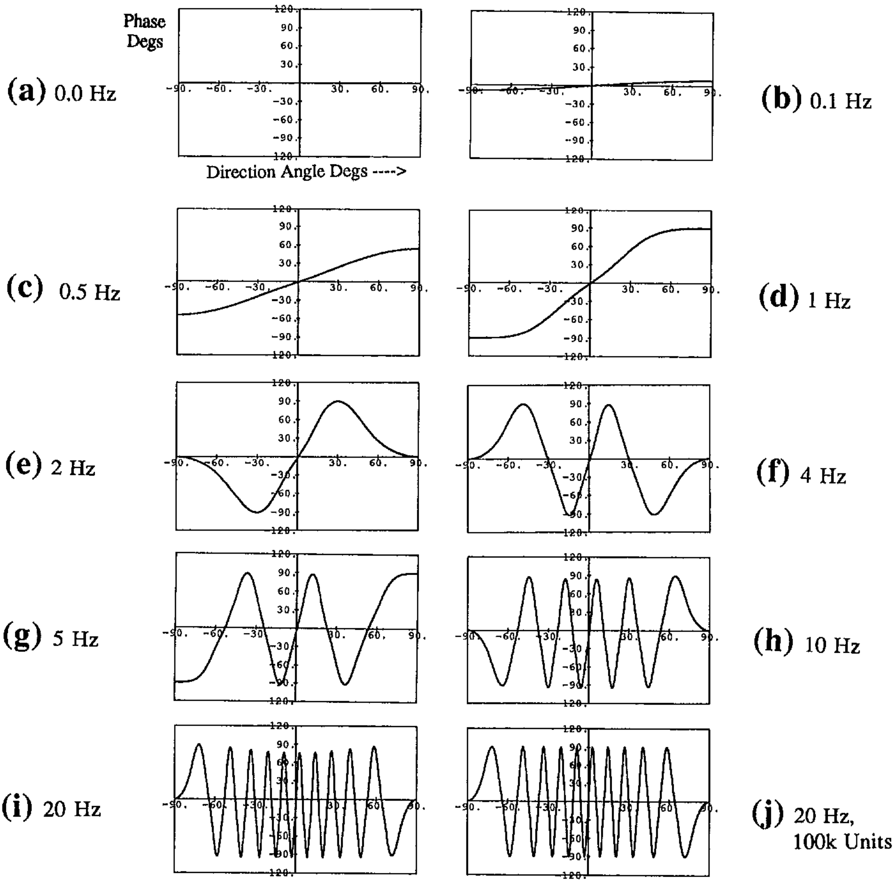


Fig. 21. Phase polar responses (phase vs direction angle) for the five-source unit-length Bessel array, at a constant distance of 20 units at frequencies of (a) 0, (b) 0.1, (c) 0.5, (d) 1, (e) 2, (f) 4, (g) 5, (h) 10, and (i) 20 Hz. Also shown is (j) 20 Hz at a distance of 100k units. The delay effects of the working distance have been compensated for thus making the on-axis phase zero in every case. The phase curves exhibit a highly-nonlinear sinusoidal like variation of phase with angle with a peak-to-peak amplitude of ±90 degs. For a fixed angular increment, the number of oscillation cycles increases with frequency.

FIVE-SOURCE BESSEL ARRAY,
1.0 UNIT C-C LENGTH
Frequency Response vs Direction Angle, Constant Distance

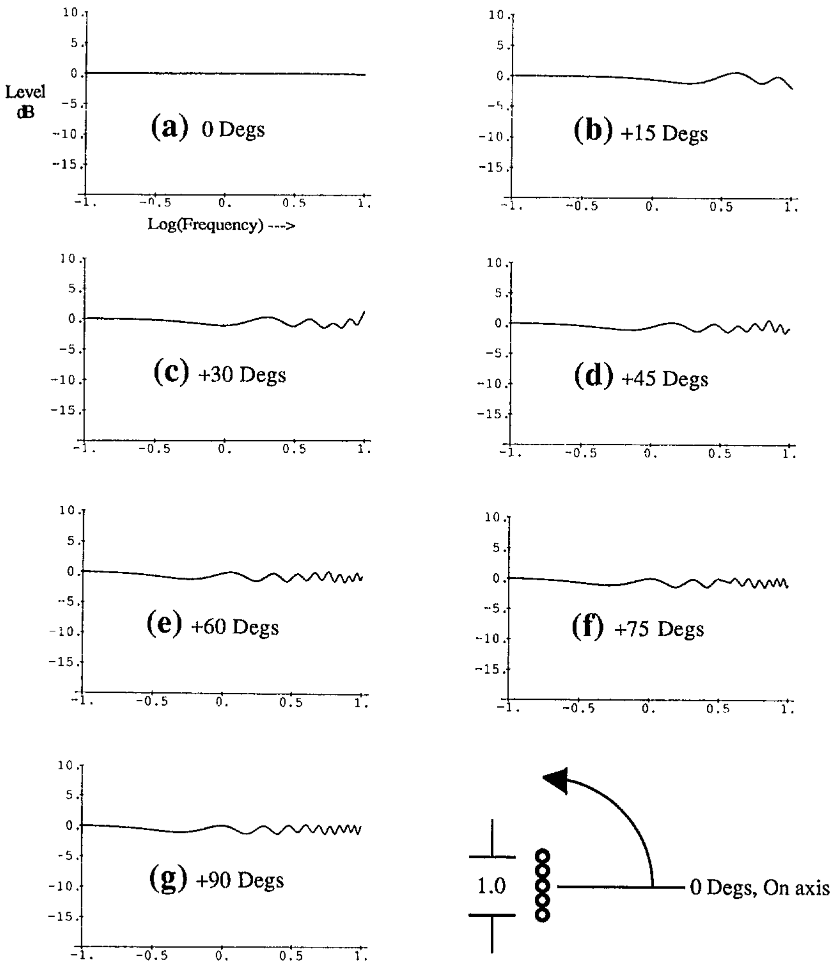


Fig. 22. Magnitude vs frequency responses for the five-source unit-length Bessel array, at a constant working distance of 20 units. The responses cover the range from 0.1 to 10 Hz, and are shown at angles ranging from 0 to +90 degs with steps of 15 degs. (a) 0 degs. (b) 15 degs. (c) 30 degs. (d) 45 degs. (e) 60 degs. (f) 75 degs. (g) 90 degs. Unlike the previous equal-level arrays, the ripple does not continually increase with angle. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.).

FIVE-SOURCE BESSEL ARRAY, 1.0 UNIT C-C LENGTH

Frequency Response vs Distance, Constant Angle

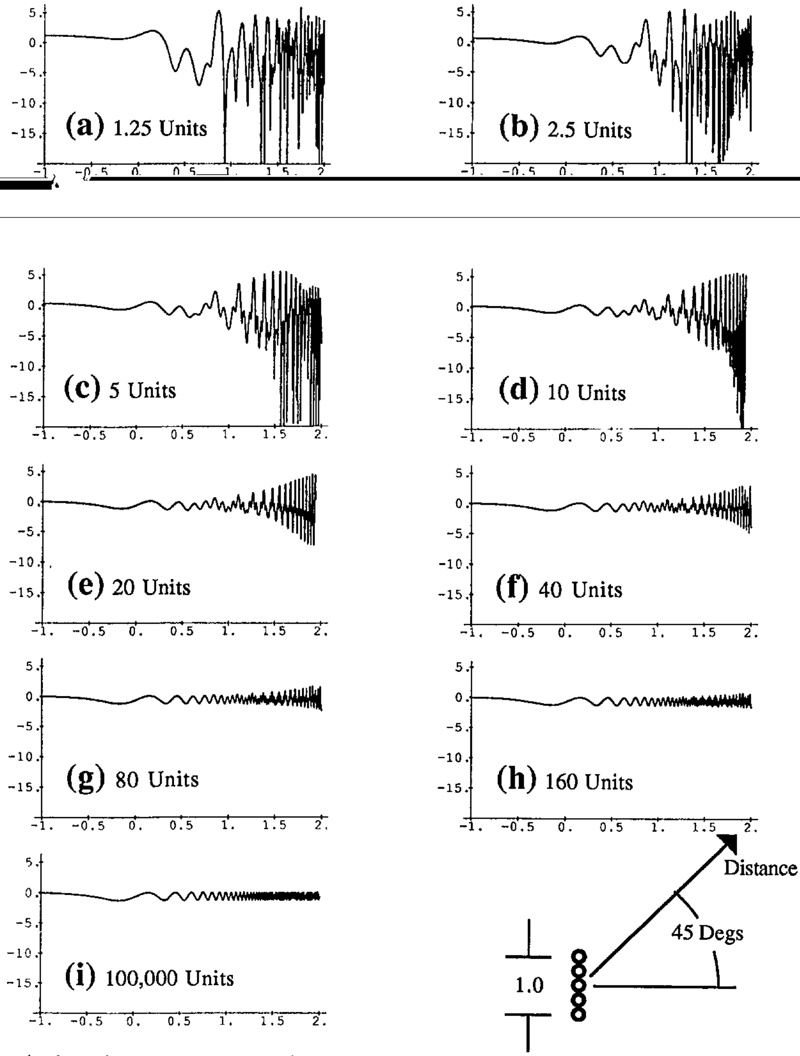


Fig. 23. Magnitude vs frequency responses for the five-source unit-length Bessel array, at a fixed angle of 45 degs and at various working distances, covering the range of 1.25 to 160 units with 1:2 steps and at 100k units. ***Note the wider frequency range of 0.1 to 100 Hz.*** (a) 1.25 units. (b) 2.5 units. (c) 5 units. (d) 10 units. (e) 20 units. (f) 40 units. (g) 80 units. (h) 160 units. (i) 100k units. Unlike the previous arrays, that the frequency response ripple continually decreases with distance until about a 2 dB p-p ripple is attained. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.).

FIVE-SOURCE BESSEL ARRAY,
1.0 UNIT C-C LENGTH
Magnitude, Phase, and Group Delay vs Frequency, at an Angle of +45 Degs and a Distance of 20 Units

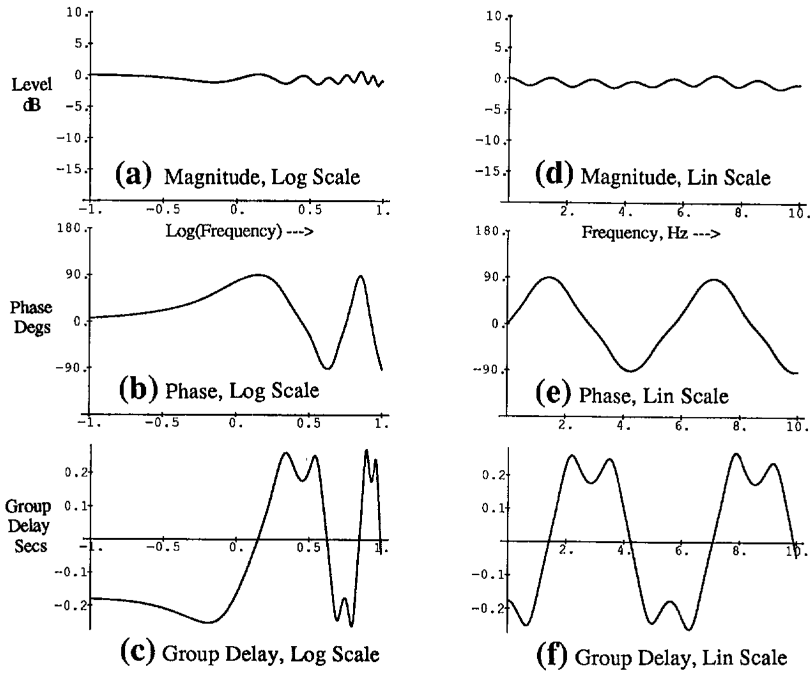
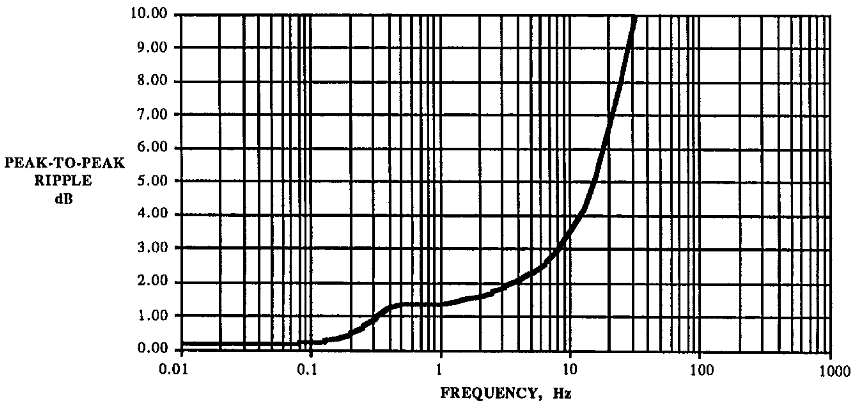


Fig. 24. Magnitude, phase, and group delay vs frequency responses for the five-source unit-length Bessel array, at 45 degs off axis at a working distance of 20 units. Both logarithmic and linear frequency scale plots are shown in this graph, up to a frequency of 10 Hz. (a) Magnitude, log scale. (b) Phase, log scale. (c) Group delay, log scale. (d) Magnitude, lin scale. (e) Phase, lin scale. (f) Group delay, lin scale. The magnitude response is mostly flat, with about a 2 dB p-p ripple. The phase varies non-linearly, in a somewhat sinusoidal manner with frequency, oscillating between ±90 degs, which indicates a non-minimum phase response. The group delay plot indicates an effective oscillatory peak shift of acoustic position of about ±25% the length of the array, as the frequency is increased!

FIVE-SOURCE BESSEL ARRAY, 1.0 UNIT C-C LENGTH
POLAR PEAK-TO-PEAK RIPPLE
vs
FREQUENCY
(Distance = 20 Units)



(Velocity of Propagation = 1.0 Unit/Sec)

Fig. 25. Polar magnitude peak-to-peak ripple in dB vs frequency for the five-source unit-length Bessel array, at a working distance of 20 units. Observe that the ripple increases much more gradually with increasing frequency as compared to the equal-level arrays. Note also the much extended bandwidth of operation as compared to the previous arrays.

FIVE-SOURCE BESSEL ARRAY, 1.0 UNIT C-C LENGTH POLAR PEAK-TO-PEAK RIPPLE vs FREQUENCY (Distance = 2.5, 5, 10, 20, 40, 80, 160, 320, 1000, 10000, 100000 Units)

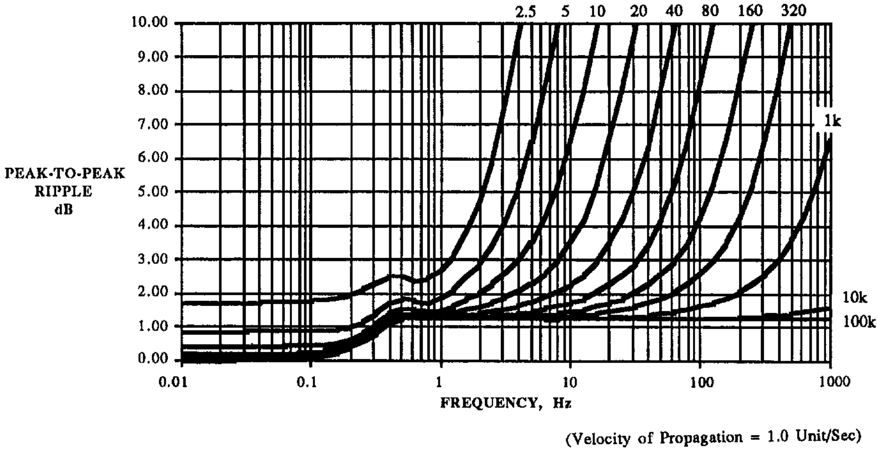


Fig 26. Polar magnitude peak-to-peak ripple in dB vs frequency for the five-source unit-length Bessel array, at a working distances covering the range of 2.5 to 100k units. It is quite evident that the operation of the Bessel array improves in direct proportion to the working distance away from the array.

FIVE-SOURCE BESSEL ARRAY, 1.0 UNIT C-C LENGTH MAXIMUM OPERATING FREQUENCY vs WORKING DISTANCE

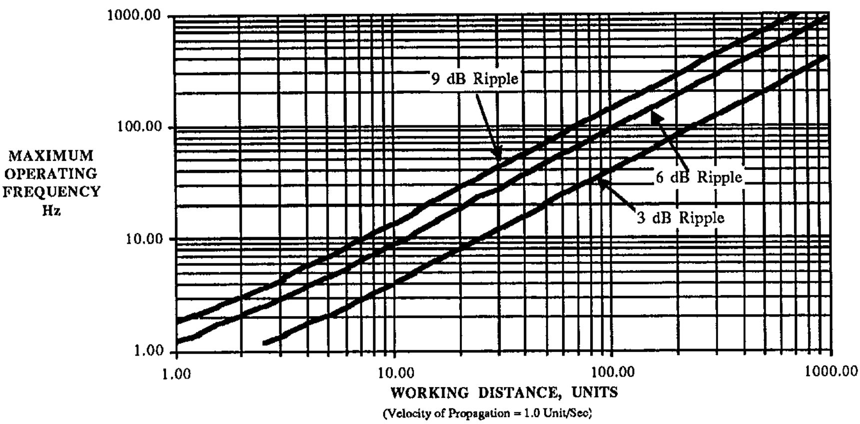


Fig. 27. Plot of maximum operating frequency vs working distance for the five-source unit-length Bessel array. Contours of equal p-p ripple at values of 3, 6, and 9 dB are shown on the graph. The direct relationship between maximum frequency and operating distance is clearly shown.

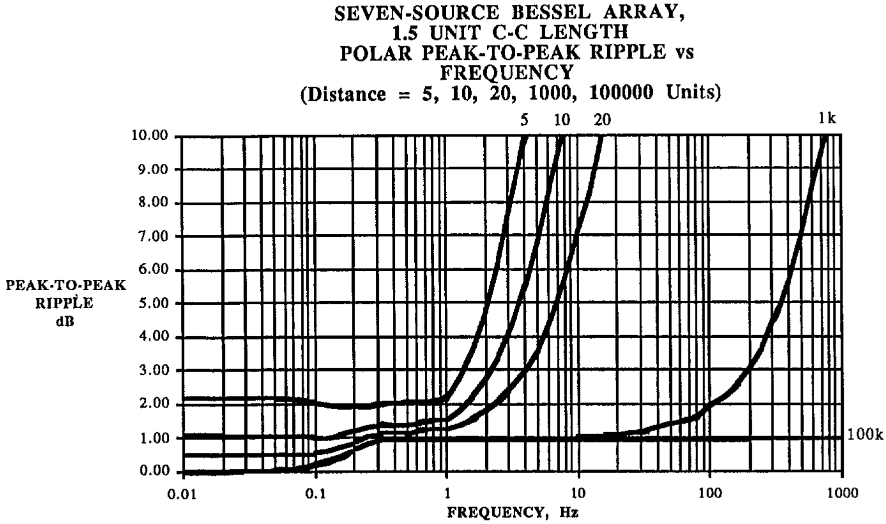


Fig. 28. Polar magnitude peak-to-peak ripple in dB vs frequency for the seven-source Bessel array, of 1.5 unit length, at working distances covering the range of 5 to 100k units. At large distances, the ripple attains a minimum plateau value of about 1.0 dB. In general, the curves are shifted to the left, as compared to the five-source Bessel, which indicates lower frequencies of operation.

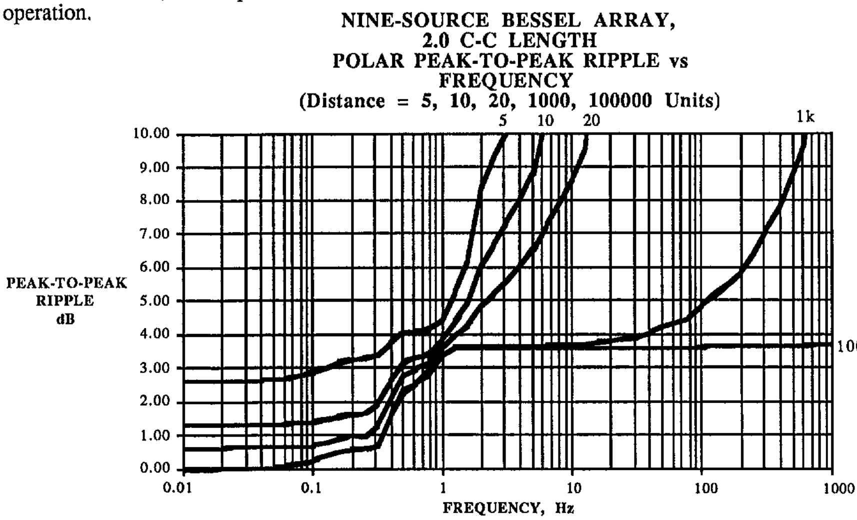


Fig. 29. Polar magnitude peak-to-peak ripple in dB vs frequency for the nine-source two-unit-length Bessel array at working distances covering the range of 5 to 100k units. At large distances, the ripple attains a minimum plateau value of about 3.6 dB, which is significantly higher than the previous Bessel arrays. As noted for the seven-source Bessel array, the curves are shifted even more to the left, as compared to the five-source Bessel, which indicates an even lower bandwidth of operation.

POLAR COMPARISON OF ALL ANALYZED ARRAYS
at a Frequency of 10 Hz and Distance of 20 Units

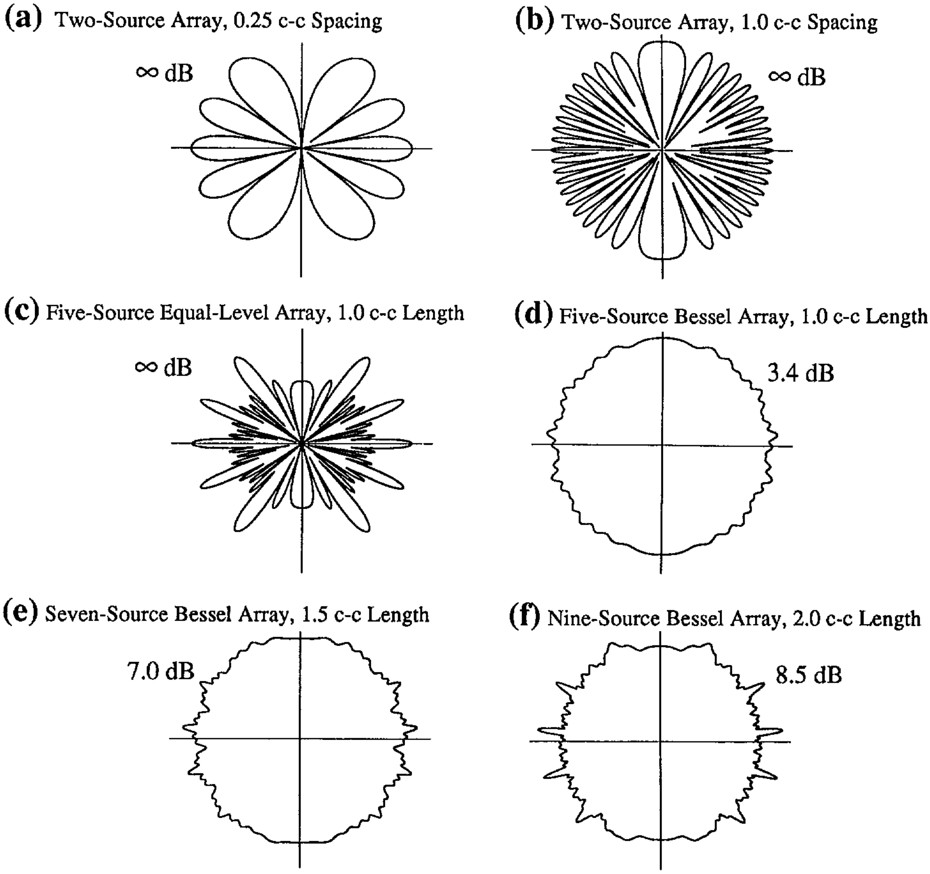


Fig. 30. Comparison of magnitude polars for all the analyzed arrays. The polars were all run at a frequency of 10 Hz and a working distance of 20 units. The peak-to-peak polar ripple is listed on each plot.
(a) Two-source equal-level equal-polarity equal-spaced array of 0.25 unit spacing.
(b) Two-source equal-level equal-polarity equal-spaced array of 1.0 unit spacing.
(c) Five-source equal-level equal-polarity equal-spaced array with 1.0 unit c-c length.
(d) Five-source Bessel array with 1.0 unit c-c length.
(e) Seven-source Bessel array with 1.5 unit c-c length.
(f) Nine-source Bessel array with 2.0 unit c-c length.
The superiority of the five-source Bessel (d) is very clear.

FREQUENCY RESPONSE COMPARISON OF ALL ANALYZED ARRAYS at an Angle of +45 Degs and Distance of 20 Units

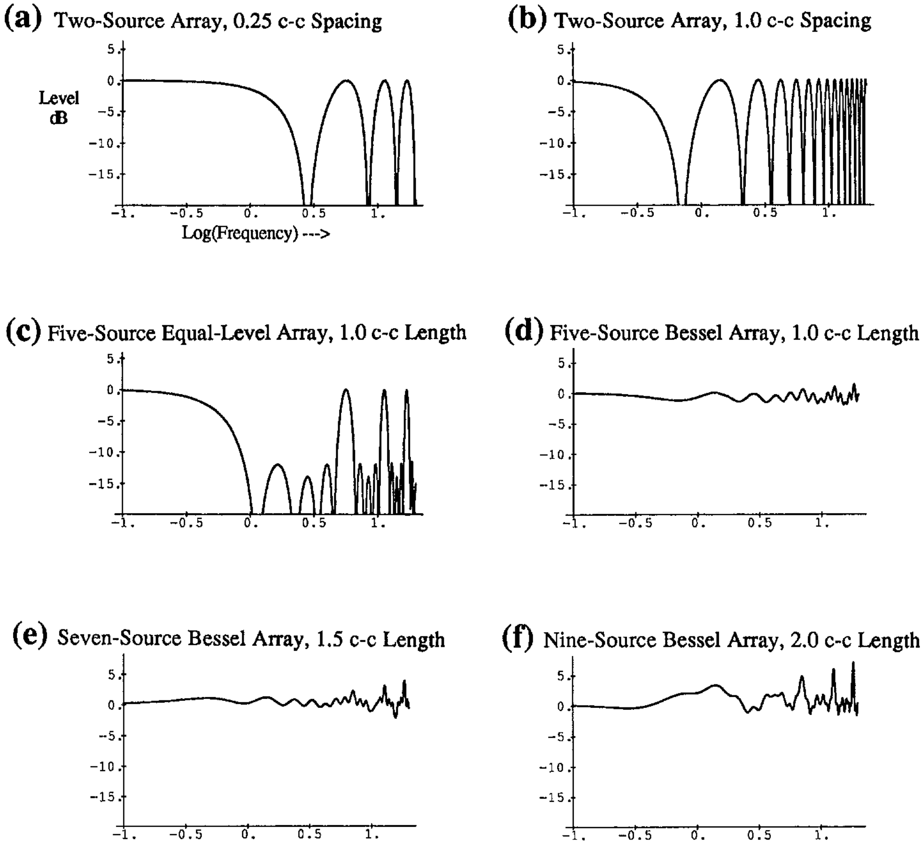


Fig. 31. Comparison of off-axis magnitude frequency responses for all the analyzed arrays. The response curves were all run at +45 degs with a working distance of 20 units, and cover the same frequency range of 0.1 to 20 Hz. Note that the log of the frequency is indicated (-1 = 0.1 Hz, 0 = 1 Hz etc.).
(a) Two-source equal-level equal-polarity equal-spaced array of 0.25 unit spacing.
(b) Two-source equal-level equal-polarity equal-spaced array of 1.0 unit spacing.
(c) Five-source equal-level equal-polarity equal-spaced array with 1.0 unit c-c length.
(d) Five-source Bessel array with 1.0 unit c-c length.
(e) Seven-source Bessel array with 1.5 unit c-c length.
(f) Nine-source Bessel array with 2.0 unit c-c length.
Again, the five-source Bessel array has the smoothest and most extended response.

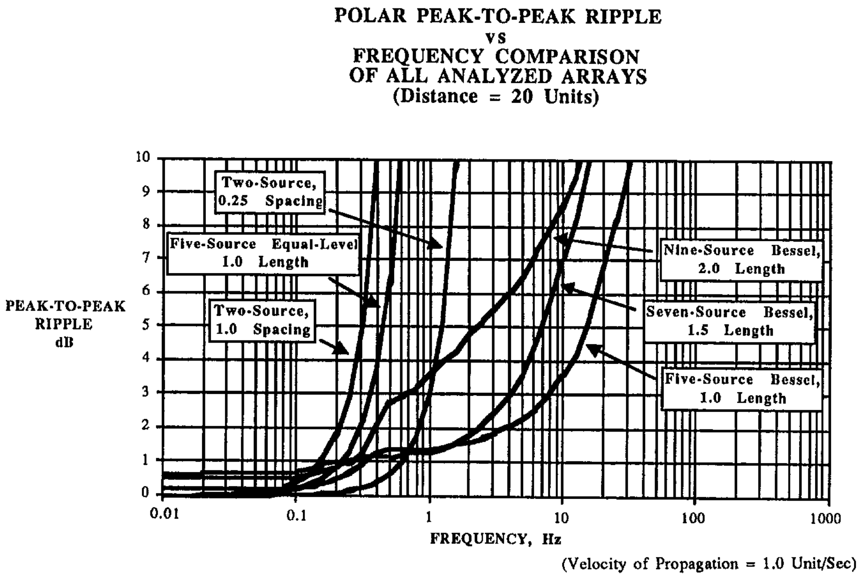


Fig. 32. Comparison of polar magnitude peak-to-peak ripple vs frequency for all the analyzed arrays, at a working distance of 20 units. The superiority of the five-source Bessel is again, quite clear.